MATH 109 - MIDTERM #2

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ANSWERS TO THE TRUE/FALSE QUESTIONS DO NOT NEED TO BE JUSTIFIED. A CORRECT ANSWER IS WORTH 5 POINTS, AN INCORRECT ANSWER IS WORTH 0 POINTS, AND A BLANK ANSWER IS WORTH 2 POINTS.

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED.

THE EXAM CONSISTS OF 4 TRUE/FALSE QUESTIONS AND 3 LONGER FORMAT QUESTIONS. YOUR ANSWERS TO THE LONGER FORMAT QUESTIONS SHOULD BE CAREFULLY JUSTIFIED. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE, BUT THEY SHOULD BE CLEARLY REFERENCED. FOR EXAMPLE,

“We prove the statement by induction on ...”
1. (20 points) Label the following statements as true or false. Any ambiguous answer (for example, resembling a hybrid of T and F) will be treated as an incorrect answer.

(a) **F** If \( f : X \to Y \) is surjective and \( A \subseteq X \), then \( f|_A : A \to Y \) is surjective.

Consider the identity function \( \text{id}_Z : Z \to Z \).

Then for any proper subset \( A \subseteq Z \),

\[ (f|_A) : A \to Z \]

is not surjective.

(b) **F** Let \( f : X \to Y \) and \( g : Y \to Z \) be functions. If \( f \) is surjective and \( g \) is injective, then \( g \circ f \) is injective.

Let \( C : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) \( x \mapsto x^2 \)

Then \( C \) is surjective

\( g \) is injective

But \( g \circ C = C \) is not injective.
(c) \[ F \] Suppose that \( X \) is a set with a proper subset \( Y \subseteq X \) such that \(|Y| < |X|\). Then \( X \) is a finite set.

Let \( X = \mathbb{R} \)
\[
Y = \mathbb{Z} \cup \{0, 3\}.
\]

(d) \[ T \] Assume that \( A \) and \( B \) are finite sets with \(|A| = n\), \(|B| = m\), and \(|A \cup B| = \max(m, n)\). Then \( A \cap B = B \) or \( A \cap B = A \).

\[ |A \cup B| = |A| + |B| - |A \cap B|. \]

Since \( |A \cup B| = \max(|A|, |B|) \),

we know that \( \min(|A|, |B|) = |A \cap B| \).

Since \( A \cap B \subseteq A, B \) and \( A, B \) are finite sets,

this implies \( A \cap B = A \) or \( A \cap B = B \).
2. (15 points) Suppose that $X$ and $Y$ are disjoint sets. Prove that the function
\[
\bigcup_{i=0}^{k} \left( \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \right) \to \mathcal{P}_k(X \cup Y)
\]
defined by $(A, B) \mapsto A \cup B$ is a bijection. Deduce that
\[
\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.
\]
We define a function $g : \mathcal{P}_k(X \cup Y) \to \bigcup_{i=0}^{k} \left( \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \right)$ such that $C \mapsto (c \cap X, c \cap Y)$. Since $X \cap Y = \emptyset$, $(c \cap X) \cap (c \cap Y) = c \cap X \cap Y = \emptyset$.
Moreover, since $C \in X \cup Y$, $C = (c \cap X) \cup (c \cap Y)$.
Thus, $|C| = |c \cap X| + |c \cap Y|$. Since $|C| = k$, this implies $(c \cap X, c \cap Y) \in \bigcup_{i=0}^{k} \left( \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \right)$.
In other words, our function is well-defined.
If we let $f$ be the function defined by the problem, then $g \circ f (A, B) = g(A \cup B) = (A \cup B) \cap X, (A \cup B) \cap Y)$
\[= (A \cap X) \cup (B \cap X), (A \cap Y) \cup (B \cap Y))
\[= (A \cup \emptyset, \emptyset \cup B) = (A, B).
\]
Similarly, $\circ g (C) = ((C \cap X, C \cap Y) = (C \cap X) \cup (C \cap Y))$
\[= C \quad \text{since } C \in X \cup Y.
\]
Thus, $f$ is a bijection. If $|X| = m$ and $|Y| = n$, then our bijection proves that $\binom{m+n}{k} = |\mathcal{P}_k(X \cup Y)| = \bigcup_{i=0}^{k} |\mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)| = \sum_{i=0}^{k} |\mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)| = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}$.
3. (15 points) Suppose that $f : A \to B$ is a function with $A$ an infinite set and $B$ a finite set. Prove that there exists an element $b \in B$ whose preimage is an infinite set.

Recall that $A = \bigcup_{b \in B} f^{-1}(\{b\})$ where

$\bigcap_{b \in B} f^{-1}(\{b\}) = \emptyset$ if $b \neq b'$.

Suppose, for a contradiction, that $f^{-1}(\{b\})$ is a finite set for every $b \in B$.

Since $B$ is finite, we can write $B = \{b_1, \ldots, b_n\}$, where $|B| = n$. Then $|A| = \left| \bigcup_{b \in B} f^{-1}(\{b\}) \right|$

$= \left| \bigcup_{i=1}^{n} f^{-1}(\{b_i\}) \right|$

$= \sum_{i=1}^{n} \left| f^{-1}(\{b_i\}) \right| < \infty$

since we assumed that $f^{-1}(\{b_i\})$ is a finite set for each $b_i$.

Thus, $A$ is a finite set, a contradiction.

We conclude that $\exists b \in B : f^{-1}(\{b\})$ is an infinite set.
4. (15 pts) Let \( n \in \mathbb{Z} \). Prove that

\[ \gcd(21n + 4, 14n + 3) = 1. \]

Let \( a = 21n + 4 \) and \( b = 14n + 3 \).

Then \( a - b = 7n + 1 \)

and \( b - 2(a - b) = 1 \).

So, \(-2a + 3b = 1\).

Thus, \( \gcd(a, b) = 1 \).

Here, we are using the fact that

\[ \gcd(a, b) = \min \left\{ \varepsilon m + bn : m, n \in \mathbb{Z} \setminus \mathbb{N} \right\}. \]
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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