REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED.

THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE, BUT THEY SHOULD BE CLEARLY REFERENCED. FOR EXAMPLE,

“BY THE PIGEONHOLE PRINCIPLE, ...”
1. (25 points) Suppose that $A$ is a subset of $[5n] = \{1, 2, \ldots, 5n\}$ for some $n \geq 1$. If $\#(A) \geq 4n + 1$, prove that $A$ necessarily has a subset of five consecutive integers, i.e., there exists an integer $m \in [5n - 4]$ such that \{m, m + 1, m + 2, m + 3, m + 4\} \subseteq A.$
2. (25 points) Let $n$ be a non-negative integer. Prove that
\[\sum_{A \subseteq [n]} (-2)^{\#(A)} = (-1)^n,\]
where the sum is over all subsets $A$ of $[n]$. 
3. (25 points) Recall that $S(n, k)$ denotes a Stirling number of the second kind. Find a closed-form expression for $S(n, n - 2)$ for $n \geq 3$. By a closed-form expression, we mean a simple formula that can be evaluated to obtain the desired quantity. For example, recall that $S(n, 2) = 2^{n-1} - 1$. In particular, simply stating the recursive identity for the Stirling numbers of the second kind will not receive any credit.
4. (25 points) Let $f$ be an $n$-permutation for some $n \geq 2$. Suppose that $i, j \in [n]$ are distinct and that $i$ and $j$ are in different cycles of $f$. Prove that $f \circ (i, j)$ has one fewer cycle than $f$. Recall that $f \circ (i, j)$ is a composition of permutations, where $(i, j)$ is the permutation that maps $i$ to $j$, $j$ to $i$, and fixes every other number. Also, recall that $f \circ (i, j)$ means apply the permutation $(i, j)$ first and then the permutation $f$.

Hint: consider the cycles $c_i$ and $c_j$ that contain $i$ and $j$ respectively.
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)
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