REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

THIS EXAM WILL BE SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED.

THE EXAM CONSISTS OF 4 QUESTIONS. YOUR ANSWERS SHOULD BE CAREFULLY JUSTIFIED. YOU ARE ALLOWED TO USE RESULTS FROM THE TEXTBOOK, HOMEWORK, AND LECTURE, BUT THEY SHOULD BE CLEARLY REFERENCED. FOR EXAMPLE,

“BY THE COMPOSITION FORMULA FOR OGFS, ...”
1. (25 points) Let \( p = p_1p_2 \cdots p_n \) be an \( n \)-permutation in one-line notation. Recall that every permutation has an inverse \( p^{-1} : [n] \to [n] \), which satisfies \( p \circ p^{-1} = p^{-1} \circ p = 12 \cdots n \). An *inversion* of \( p \) is a pair of entries \((p_i, p_j)\) such that \( i < j \) and \( p_i > p_j \) (so \( p \) inverts the order of \( i \) and \( j \)). Suppose that \( p \) has exactly \( k \) inversions. Prove that \( p^{-1} \) also has exactly \( k \) inversions.
2. (25 points) Let \( p_{[j,k]}(n) \) be the number of partitions of \( n \) whose largest part is equal to \( k \) and whose smallest part is equal to \( j \) (assume that \( j < k \)). Decide whether you want to find the ordinary generating function
\[
\sum_{n=0}^{\infty} p_{[j,k]}(n)x^n
\]
or the exponential generating function
\[
\sum_{n=0}^{\infty} p_{[j,k]}(n)\frac{x^n}{n!}.
\]
Once you have made your choice, make sure to actually find the corresponding generating function.
3. (25 points) Suppose that we have a bookcase with two shelves: one top shelf and one bottom shelf. Let $b_n$ be the number of ways to partition a set of $n$ distinct books into two non-empty groups and then line up one group on the top shelf and the other group on the bottom shelf. Decide whether you want to find the ordinary generating function

$$\sum_{n=0}^{\infty} b_n x^n$$

or the exponential generating function

$$\sum_{n=0}^{\infty} b_n \frac{x^n}{n!}.$$ 

Once you have made your choice, make sure to actually find the corresponding generating function and use it to find a closed-form expression for $b_n$. 


4. (25 points) Let $G = (V, E)$ be a finite graph with distinct vertices $v, w \in V$. Suppose that there is a walk from $v$ to $w$ in $G$. Prove that there is also a path from $v$ to $w$ in $G$. 
(ADDITIONAL SPACE FOR WORK, CLEARLY INDICATE THE PROBLEM YOU ARE WORKING ON)