REVIEWS QUESTIONS

- Cumulative exam with emphasis on material since the second midterm. Chapters 1, 3, 4, 5, 6, 7, 8, 9, 10. At most nine questions on the exam. Not necessarily one for each chapter, though you might be asked questions from earlier chapters. Generating functions are especially important. Graphs are especially important.
- Spanning trees of a graph in terms of a smaller graph. $G \setminus \{e\}$. $G/\sim e$.
- Let $G$ be a bipartite graph. Let $A$ be the adjacency matrix of $G$. Prove that the matrix $A^k$ has zeroes on the diagonal for $k$ odd. Prove that a bipartite graph has no cycles of odd length.
- Exponential generating function $F(x)$ for the number $f_n$ of forests on $[n]$ whose components have size $\leq 3$. $1, 1, 3$. Trees on $[n]$ with vertex size 1? $x$. Trees on $[n]$ with vertex size 2? $x^2/2$. Trees on $[n]$ with vertex size 3? $x^3/2$. Exponential formula tells us $e^x e^{x^2/2} e^{x^3/2}$.
- Let $G(x)$ be the exponential generating function for the numbers $g_n$ of all rooted trees on the vertex set $[n]$. Prove that $G(x) = xe^{G(x)}$.
- Try to use composition formula. No rooted trees with zero vertices, $g_0 = 0$. Root means there is actually a vertex to select.
- Draw picture. Clearly there’s a recursive structure. How to do this rigorously? Let $h_n$ be the number of rooted forests on $[n]$.
- $e^G(x) = \sum_{n=0}^{\infty} h_n x^n$ gives the number of rooted forests on $[n]$. We set $h_0$ by convention because of the compositional formula. How is the number $g_n$ related to $h_n$?
- $g_{n+1} = (n+1)h_n$ for all $n \geq 0$. This still makes sense for $n = 0$ because of the convention that $h_0 = 1$.
- $G(x) = \sum_{n=1}^{\infty} g_n x^n = \sum_{n=0}^{\infty} g_{n+1} x^{n+1} = xe^{G(x)}$. Can interpret using multiplication of exponential generating functions, but I find this to be more subtle to explain. On the exam, use what works.
- Let $n > k > 0$ be integers. Show that the number of $n$-permutations with exactly $k$ cycles and with 1 and 2 in the same cycle is the same as the number of permutations of $[n]$ with $k+1$ cycles and with 1 and 2 in different cycles.
- Coloring $n$ tiles one of the colors red, blue, green with the requirement that the green tiles should all be to the left of the blue tiles. If no green, then $2^n$. If last green at position $k$, then right of it is $2^{n-k}$ for blue/red and left of it is $2^{k-1}$ for green/red. $\sum_{k=1}^{n} 2^{n-k} 2^{k-1} = n2^{n-1}$. $n2^{n-1} + 2^n$.
- Suppose that $G$ is a simple graph with $n$ vertices. What is the maximum number of edges that $G$ can have?
- A pair partition of $[n]$ is a partition $\pi$ of $[n]$ such that each block of $\pi$ has size 2. Thus, each block is a pairing of numbers. Determine the number of pair partitions on $n$.
- A non-crossing pair partition is a pair partition without any crossing. Prove that the number of non-crossing pair partitions of $[2n]$ is equal to the Catalan number $C_n$.
- Let $a_n$ count the number of ways to divide a group of $n$ people into three subgroups $A, B, C$ and ask each subgroup to form a line. We also require that $A$ have an odd number of people and $B$ have an even number of people. Find the appropriate generating function.