Today: ASV 1.5 (Random variables)

ASV 3.1 (Probability distributions)

Next: ASV 3.2

Week 2: Homework 1 (due Friday, Jan 15)
E.g. (from last lecture)  

An urn has 4 red and 7 blue balls. Choose two balls.  

\[ A = \{1^{st} \text{ ball is red}\} \]  
\[ B = \{2^{nd} \text{ ball is blue}\} \]

1) choose balls with replacement  
\[ P(A) = \frac{4 \cdot 11}{11 \cdot 11} = \frac{4}{11} \]  
\[ P(B) = \frac{11 \cdot 7}{11 \cdot 11} = \frac{7}{11} \]  
\[ P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 11} = P(A)P(B) \]  
\[ \text{A and B independent} \]

2) choose balls without replacement  
\[ P(A) = \frac{4 \cdot 10}{11 \cdot 10} = \frac{4}{11} \]  
\[ P(B) = \frac{10 \cdot 7}{11 \cdot 10} = \frac{7}{11} \]  
\[ P(A \cap B) = \frac{4 \cdot 7}{11 \cdot 10} \neq \frac{4 \cdot 7}{11 \cdot 11} \]  
\[ \text{A and B are not independent} \]
A and B independent $\iff$ A and $B^c$ independent

**Proof.** $(\Rightarrow)$ Suppose that A and B are independent.

\[
P(A \cap B^c) = P(A) - P(A \cap B) = P(A) - P(A)P(B) = P(A)(1 - P(B))
\]

\[
= P(A)P(B^c)
\]

\[
A = (A \cap B) \cup (A \cap B^c)
\]

& disjoint

\[
P(A) = P(A \cap B) + P(A \cap B^c)
\]
More than two events?

**Def.** A collection \( A_1, \ldots, A_n \) of events is mutually independent if

for any subcollection \( A_{i_1}, A_{i_2}, \ldots, A_{i_k} \) \((1 \leq i_1 < i_2 < \cdots < i_k \leq n)\)

\[
P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k})
\]

**E.g.** When \( n = 3 \), this means that we must have

\[
\begin{align*}
P(A_1 \cap A_2) &= P(A_1) P(A_2) \\
P(A_1 \cap A_3) &= P(A_1) P(A_3) \\
P(A_2 \cap A_3) &= P(A_2) P(A_3) \\
P(A_1 \cap A_2 \cap A_3) &= P(A_1) P(A_2) P(A_3)
\end{align*}
\]
Important example

Toss a coin three times

\[ A = \{ \text{there is exactly 1 Tails in the first two} \} \]

\[ B = \{ \text{there is exactly 1 Tails in the last two} \} \]

\[ C = \{ \text{there is exactly 1 Tails in first and last tosses} \} \]

\[ A = \{ (H, T, \ast), (T, H, \ast) \} \quad B = \{(*, H, T), (*, T, H)\} \]

\[ C = \{ (H, \ast, T), (T, \ast, H) \} \]

\[ P(A) = \frac{4}{8} = \frac{1}{2} = P(B) = P(C) \]

\[ P(A \cap B) = \frac{2}{8} = \frac{1}{4} = P(A \cap C) = P(B \cap C) \]

\[ A \cap B \cap C = \emptyset \]

\[ P(A \cap B \cap C) = 0 \]

\[ \Downarrow \quad A, B, C \text{ are pairwise indep.} \]
**Random variables**

\((\Omega, \mathcal{F}, P)\) - probability space

**Definition.** A (measurable*) function \(X : \Omega \rightarrow \mathbb{R}\) is called a random variable.

\[
\{w \in \Omega : X(w) \in B\} = \{X \in B\} \subset \Omega \quad \text{(event)}
\]

For any \(B \subset \mathbb{R}\) we can compute \(P(X \in B)\).
**Definition.** Let $X$ be a random variable (rv). The probability distribution of $X$ is the collection of probabilities $P(X \in B)$ for all $B \subseteq \mathbb{R}$.

**Remark.** Strictly speaking, $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a Borel measurable function.

**Examples**

1) Coin toss: $\Omega = \{H, T\}$, $X(H) = 0$, $X(T) = 1$

   $P(X = 0) = P(\{H\}) = \frac{1}{2} = P(X = 1)$ (fair coin)

2) Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $X(w) = w$

   For any $1 \leq i \leq 6$, $P(X = i) = \frac{1}{6}$
3) Roll a die twice: \( \Omega = \{(i,j): i,j \in \{1, \ldots, 6\}\} \)

\( X_1((i,j)) = i \) (first number) \( X_2((i,j)) = j \) (second number)

for \( 1 \leq i \leq 6 \) \( P(X_1 = i) = \frac{1}{6} \) \( P(X_2 = i) = \frac{1}{6} \)

\[ S = X_1 + X_2 \]

\[ P(S = 2) = \frac{1}{36} \]
\[ P(S = 3) = \frac{2}{36} \]
\[ P(S = 4) = \frac{3}{36} \]
\[ P(S = 5) = \frac{4}{36} \]
\[ P(S = 6) = \frac{5}{36} \]
\[ P(S = 7) = \frac{6}{36} \]
\[ P(S = 8) = \frac{5}{36} \]
\[ P(S = 9) = \frac{4}{36} \]
\[ P(S = 10) = \frac{3}{36} \]
\[ P(S = 11) = \frac{2}{36} \]
\[ P(S = 12) = \frac{1}{36} \]
4) Choosing a point from unit disk uniformly at random

\[ \Omega = \{ \mathbf{w} \in \mathbb{R}^2 : \text{dist}(\mathbf{w}, 0) \leq 1 \} \]

\[ X(\mathbf{w}) = \text{dist}(\mathbf{w}, 0) \]

For any \( r < 0 \), \( P(X \leq r) = 0 \)
For any \( r > 1 \), \( P(X \leq r) = 1 \)

For any \( r \in [0, 1] \), \( P(X \leq r) = \frac{\text{size } D_r}{\text{size } D_1} = \frac{\pi r^2}{\pi} = r^2 \)

\[ \{ X \leq r \} = \{ X \in (-\infty, r] \} \]

\( \text{missing in class} \)
Def. Random variable $X$ is a discrete rv if there exists a finite or infinite countable collection of points \( \{a_i \} \subset \mathbb{R} \) such that \( \sum_{i=1}^{\infty} P(X = a_i) = 1 \)

Example (lecture 3) Toss a coin until first $T$.

\( X \) = total number of tosses.
(Already computed before) for any $i = 1, 2, \ldots$

\[
P(X = i) = \frac{1}{2^i}
\]

\[
\sum_{i=1}^{\infty} P(X = i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \quad \text{(geometric series)}
\]
Discrete rv $X$ is completely described by its probability mass function (pmf) $p_X$ given by

$$p_X(k) = P(X=k)$$

for all possible values of $X$.

\[Ex.\  S = \text{sum of two dice}\]

\[
\begin{array}{cccccccccccc}
  k & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
  p_S(k) & \frac{1}{36} & \frac{2}{36} & -- & -- & & & & & & & \\
\end{array}
\]

What if for every $x \in \mathbb{R}$ $P(X=x)=0$?
Probability density function

Def. Let $X$ be a rv. If function $f: \mathbb{R} \to \mathbb{R}$ satisfies

$$P(X \leq b) = \int_{-\infty}^{b} f(x) \, dx$$

then $f$ is a probability density function of $X$

Remark. Definition implies that for $B \subset \mathbb{R}$

$$P(X \in B) = \int_{B} f_{X}(x) \, dx$$
E.g. Distance to 0 from a random point in a disk

\[ \int_{-\infty}^{\infty} f_X(x) \, dx = P(X \leq r) = \begin{cases} 0, & r > 0 \\ r^2, & 0 \leq r \leq 1 \\ 1, & r > 1 \end{cases} \]

\[ f_X(x) = \]