Today: Expectation

Next: ASV 3.4

Week 4:

- Homework 3 (due Sunday, January 31)
**Poisson Distribution**

A random variable $X$ has the Poisson($\lambda$) distribution if

$$\lim_{n \to \infty} P(S_n \geq k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^\lambda = 1 \quad \checkmark$$

E.g. A 100 year storm is a storm magnitude expected to occur in any given year with probability $1/100$.

Over the course of a century, how likely is it to see at least 4 100 year storms?

$$P(S_{100, 1/100} \geq 4) = \sum_{k=4}^{100} P(S_{100, 1/100} = k) \approx \sum_{k=4}^{100} e^{-1} \frac{(1)^k}{k!}$$

$$\sum_{k=4}^{100} \left(\frac{1}{100}\right)^k \left(1 - \frac{1}{100}\right)^{100-k} \approx 1.8374 \quad \left(= \sum_{k=4}^{100} e^{-1} \frac{1}{k!} \approx 0.18374\right)$$

$$\sum_{k=0}^{3} e^{-1} \frac{1}{k!} = 1 - \sum_{k=0}^{3} e^{-1} \frac{1}{k!} = 1.8288\%$$
Summary

Sampling independent trials, the most important (discrete) probability distributions are:

- **Ber(p)**: \( P(X=1) = p, \ P(X=0) = 1-p \quad 0 \leq p \leq 1 \)
  (single trial with success probability \( p \))

- **Bin(n,p)**: \( P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad 0 \leq k \leq n \)
  (number of successes in \( n \) independent trials with rate \( p \))

- **Geom(p)**: \( P(N=k) = (1-p)^{k-1} p \quad k=0,1,2,... \)
  (first successful trial in repeated independent trials with rate \( p \))

- **Poisson(\lambda)**: \( P(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \quad k=0,1,2,... \quad \lambda > 0 \)
  (Approximates \( \text{Bin}(n, \lambda/n) \); number of rare events in many trials)
Expectation

Toss a fair coin 1000 times, and record the sequence of outcomes.

Average them: $\frac{1}{1000} (1+1+0+1+0+0+0+1+1+0+1+1+1+0+0+1+1+1) \ldots$

What size do you expect this number to have?

About $\frac{1}{2}$ of the outcomes are 1, about $\frac{1}{2}$ are 0.

What if the coin is biased $\Pr(X_j=1)=p$, $\Pr(X_j=0)=1-p$.

Definition: Let $X$ be a discrete random variable with possible values $t_1, t_2, t_3, \ldots$. The expectation or expected value of $X$ is

\[ E(X) := \sum_{i} t_i \Pr(X=t_i) \]

"weighted average"
Question: Is the expectation $E(X)$ the value $X$ is equal to most often?

(a) Yes, always.
(b) No, not generally.

Eg. Let $X$ be the number rolled on a fair die. $X \in \{1,2,3,4,5,6\}$

$$E(X) = \sum_{k=1}^{6} k \cdot \frac{1}{6} = \frac{1}{6} (1+2+3+4+5+6) = \frac{21}{6} = \frac{7}{2}.$$ 

Eg. Let $Y$ be $\text{Ber}(p)$. $E(Y) = p \cdot 1 + (1-p) \cdot 0 = p$

Eg. You toss a biased coin $(Y)$ repeatedly until the first heads. How long do you expect it to take?

$N =$ the time the 1st heads comes up. $N \sim \text{Geom}(p)$

$$E(N) = \sum_{k=1}^{\infty} k \cdot P(N = k) = p \sum_{k=1}^{\infty} k (1-p)^{k-1} \cdot \frac{1}{1-P} = p \cdot \frac{t}{(1-(1-p))^2} = p \cdot \frac{1}{(1-x)^2}$$
E.g. \( S_n \sim \text{Bin}(n,p) \) (\( S_n = X_1 + X_2 + \ldots + X_n \) for \( X_j \) independent \( \text{Ber}(p) \))

\[
\mathbb{E}(S_n) = \sum_{k=0}^{n} k \cdot \mathbb{P}(S_n = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k} = np
\]

\( S_n = X_1 + X_2 + \ldots + X_n \)

\( \mathbb{E}(X_j) = p \)

E.g. \( X \sim \text{Poisson}(\lambda) \)

\[
\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = \lambda
\]

\( \lambda = 3 \) \( \Rightarrow \) E.g. A factory has, on average, 3 accidents per month. Estimate the probability that there will be exactly 2 accidents this month.

\( X = \#\text{accidents/month} \)

\[ \mathbb{E}(X) = 3 = \lambda \]

\[ \mathbb{P}(X=2) = e^{-3} \frac{3^2}{2!} = 0.224\% \]
E.g. Toss a fair coin until tails comes up. If this is on the first toss, you win $2 and stop. If heads comes up, the pot doubles, and you continue. That is, if the first tails is on the $k^{th}$ toss, you win $2^k$ dollars.

What is your expected winnings?

$$W = \{ 2^k \text{ if the first tails is on the } k^{th} \text{ toss} \}$$

$$E(w) = \sum_{k=1}^{\infty} 2^k \cdot P(w=2^k) = \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k} = \frac{1}{2} \sum_{k=1}^{\infty} \left( \frac{1}{2} \right)^{k-1} = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 2$$