REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

IF YOU DO NOT ASSIGN THE PAGES OF YOUR WORK TO THE QUESTIONS OF THE EXAM IN YOUR UPLOAD TO GRADESCOPE, YOU WILL LOSE POINTS (1 POINT FOR EVERY QUESTION THAT YOU FAIL TO ASSIGN THE PAGES TO).

WRITE YOUR NAME AND STUDENT ID ON THE FIRST PAGE OF YOUR UPLOAD.

WRITE YOUR SOLUTIONS TO EACH PROBLEM ON SEPARATE PAGES. CLEARLY INDICATE AT THE TOP OF EACH PAGE THE NUMBER OF THE CORRESPONDING PROBLEM. DIFFERENT PARTS OF THE SAME PROBLEM CAN BE WRITTEN ON THE SAME PAGE (FOR EXAMPLE, PART (A) AND PART (B)).

FROM THE MOMENT YOU ACCESS THE FINAL ON GRADESCOPE, YOU WILL HAVE A TOTAL OF 90 MINUTES TO COMPLETE AND UPLOAD YOUR SOLUTIONS TO GRADESCOPE. THE EXAM IS WRITTEN TO TAKE 70 MINUTES. IT IS YOUR RESPONSIBILITY TO UPLOAD YOUR SOLUTIONS ON TIME.
THE FINAL IS OPEN TEXTBOOK, AND YOU CAN REWATCH LECTURES AND REVIEW THE LECTURE SLIDES. YOU ARE NOT ALLOWED TO USE CALCULATORS, COMPUTER ALGEBRA SYSTEMS, OR SIMILAR SOFTWARE. YOU ARE NOT ALLOWED TO CONSULT OTHER PEOPLE OR RESOURCES ON THE INTERNET. IN PARTICULAR, CONTRACT CHEATING AND OTHER INSTANCES OF ACADEMIC DISHONESTY CAN RESULT IN SUSPENSION OR DISMISSAL.

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EXCEL WITH INTEGRITY PLEDGE

I pledge to be fair to my classmates and instructors by completing all of my academic work with integrity. This means that I will respect the standards set by the instructor and institution, be responsible for the consequences of my choices, honestly represent my knowledge and abilities, and be a community member that others can trust to do the right thing even when no one is watching. I will always put learning before grades, and integrity before performance. I pledge to excel with integrity.

In addition to the above, I pledge that I did not receive outside assistance with this exam. Outside assistance includes but is not limited to other people, the internet, and resources beyond the textbook, lecture notes/videos, and homework assignments.

To acknowledge that you agree to this pledge, you must copy the sentence:

I choose to excel with integrity as a member of the University of California, San Diego.

Make sure to sign your name below it and date it as well. Exams without this pledge will not be graded.
1. (20 points) Suppose that $X \sim \text{Unif}(-5, 5)$. Let $Y = |X|$.

(a) (10 points) Calculate $\text{Cov}(X, Y)$.

(b) (10 points) Are $X$ and $Y$ independent? Make sure to justify your answer.

Proof of (a). Recall that $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y]$. Since $\mathbb{E}[X] = \frac{-5 + 5}{2} = 0$, we only need to calculate $\mathbb{E}[XY] = \mathbb{E}[X|X|]$, where $X|X|$ can be thought of as a function of $X$. In particular, if $g(x) = x|x|$, then $g(X) = X|X|$. Since $X$ is a continuous random variable, we can easily calculate this expectation:

$$
\mathbb{E}[X|X|] = \mathbb{E}[g(X)] = \int_{-5}^{5} g(x) \frac{1}{10} \, dx = \int_{-5}^{5} x|x| \frac{1}{10} \, dx.
$$

Since the integrand is an odd function, this definite integral is equal to 0. So,

$$
\text{Cov}(X, Y) = 0.
$$

You could have split the integral up into

$$
\int_{-5}^{5} x|x| \frac{1}{10} \, dx = \int_{-5}^{0} -x^2 \frac{1}{10} \, dx + \int_{0}^{5} x^2 \frac{1}{10} \, dx
$$

and done the integration to get the same answer. □

Proof of (b). The covariance of $X$ and $Y$ is 0, but this does not imply that $X$ and $Y$ are independent. For example, $\mathbb{P}(X \in (3, 5)) = \frac{2}{10}$ and

$$
\mathbb{P}(Y \in (0, 2)) = \mathbb{P}(|X| \in (0, 2)) = \mathbb{P}(X \in (-2, 0) \cup (0, 2)) = \frac{4}{10},
$$

but

$$
\mathbb{P}(X \in (3, 5), Y \in (0, 2)) = \mathbb{P}(X \in (3, 5), |X| \in (0, 2))
$$

$$
= \mathbb{P}(X \in (3, 5), X \in (-2, 0) \cup (0, 2))
$$

$$
= 0
$$

$$
\neq \mathbb{P}(X \in (3, 5)) \mathbb{P}(Y \in (0, 2)).
$$

□
2. (20 points) Two people are participating in an auction to buy a painting. The auction is anonymous, meaning that the participants do not know the offer the other person submits. The person who submits the highest offer wins the auction: the value of the highest offer is called the winning bid. Suppose that participant 1 submits a random offer \( X \sim \text{Unif}(0, 1) \) and participant 2 submits a random offer \( Y \sim \text{Unif}(0, 2) \). Since the auction is anonymous, we assume that \( X \) and \( Y \) are independent.

(a) (10 points) Find the probability that participant 2 wins the auction. Hint: draw a rectangle.

(b) (10 points) Suppose that someone tells you that the winning bid was strictly greater than \( \frac{1}{2} \). Given this information, what is the probability that participant 2 won the auction? Hint: draw a rectangle.

**Proof of (a).** The problem amounts to computing \( P(X < Y) \). Since \( X \) and \( Y \) are independent, they have a joint density \( f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \). In particular, we see that \((X, Y)\) is just uniformly distributed on the rectangle \([0, 1] \times [0, 2]\). The probability \( P(X < Y) \) amounts to the area of the yellow region divided by the area of the rectangle in the following picture:

We conclude that the probability is

\[
P(X < Y) = \frac{\frac{3}{2}}{2} = \frac{3}{4}.
\]
Proof of (b). The problem amounts to computing

\[ P \left( X < Y \mid \max(X, Y) > \frac{1}{2} \right) = \frac{P \left( X < Y, \max(X, Y) > \frac{1}{2} \right)}{P \left( \max(X, Y) > \frac{1}{2} \right)}. \]

Again, since \((X, Y)\) is uniformly distributed on the rectangle \([0, 1] \times [0, 2]\), the probability

\[
\frac{P \left( X < Y, \max(X, Y) > \frac{1}{2} \right)}{P \left( \max(X, Y) > \frac{1}{2} \right)}
\]

amounts to the area of the green region divided by the area of the brown region:

We conclude that the probability is

\[
\frac{P \left( X < Y, \max(X, Y) > \frac{1}{2} \right)}{P \left( \max(X, Y) > \frac{1}{2} \right)} = \frac{11}{16} = \frac{11}{14}.
\]

\[ \square \]
3. (20 points) Suppose that $X$ and $Y$ are independent random variables with $X \sim \text{Unif}(0, 1)$ and $Y \sim \mathcal{N}(0, 1)$.

(a) (15 points) Find the density of $Z = X + Y$. You may leave your answer in terms of $\Phi$, the CDF of the standard normal distribution.

(b) (5 points) Determine the value of
$$
\int_{-\infty}^{\infty} \Phi(t + 1) - \Phi(t) \, dt.
$$
Make sure to justify your answer.

Proof of (a). We use the convolution formula:
$$
f_Z(t) = \int_{-\infty}^{\infty} f_X(x) f_Y(t - x) \, dx
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} 1_{\{t - x \in (0, 1)\}} \, dx
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} 1_{\{x - t \in (-1, 0)\}} \, dx
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} 1_{\{x \in (t - 1, t)\}} \, dx
= \int_{t-1}^{t} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx
= \Phi(t) - \Phi(t - 1).
$$

Proof of (b). Note that
$$
\int_{-\infty}^{\infty} \Phi(t + 1) - \Phi(t) \, dt = \int_{-\infty}^{\infty} \Phi(t) - \Phi(t - 1) \, dt
= \int_{-\infty}^{\infty} f_Z(t) \, dt
= 1,
$$
where the last equality follows from the fact that $f_Z(t)$ is a probability density function.
4. (20 points) Suppose that the random vector \((X_1, X_2, \ldots, X_{2r})\) has the multinomial distribution \((X_1, \ldots, X_{2r}) \sim \text{Mult}(n, 2r, p_1, p_2, \ldots, p_{2r})\). Calculate the expectation \(\mathbb{E} \left[ \sum_{i=1}^{r} X_{2i} \right]\) (note: the subscript of the \(X\) in the summation is \(2i\), not \(i\)).

**Proof.** By linearity of expectation,

\[
\mathbb{E} \left[ \sum_{i=1}^{r} X_{2i} \right] = \sum_{i=1}^{r} \mathbb{E}[X_{2i}].
\]

Since \((X_1, \ldots, X_{2r}) \sim \text{Mult}(n, 2r, p_1, p_2, \ldots, p_{2r})\), we know that \(X_j \sim \text{Bin}(n, p_j)\).

So, \(\mathbb{E}[X_{2i}] = np_{2i}\) and

\[
\mathbb{E} \left[ \sum_{i=1}^{r} X_{2i} \right] = \sum_{i=1}^{r} \mathbb{E}[X_{2i}] = \sum_{i=1}^{r} np_{2i} = n \sum_{i=1}^{r} p_{2i}.
\]

\(\square\)
5. (20 points) Suppose that we have an infinite sequence of i.i.d. random variables \( Z_1, Z_2, \ldots \) with mean \( \mathbb{E}[Z_1] = \mu \) and variance \( \text{Var}(Z_1) = \sigma^2 \). Determine the following limits with precise justifications. If appropriate, you may leave your answer in terms of \( \mu, \sigma^2 \), and \( \Phi \), the CDF of the standard normal distribution.

(a) (10 points)
\[
\lim_{n \to \infty} \mathbb{P}(Z_1 + Z_2 + \cdots + Z_n < n(\mu - 1)).
\]

(b) (10 points)
\[
\lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1 + Z_2 + \cdots + Z_n}{n} < \mu\right).
\]

Proof of (a).
\[
\lim_{n \to \infty} \mathbb{P}(Z_1 + Z_2 + \cdots + Z_n < n(\mu - 1)) = \lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1 + Z_2 + \cdots + Z_n}{n} < \mu - 1\right)
= \lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1 + Z_2 + \cdots + Z_n}{n} - \mu < -1\right)
\leq \lim_{n \to \infty} \mathbb{P}\left(\left|\frac{Z_1 + Z_2 + \cdots + Z_n}{n} - \mu\right| > 1\right)
= 0,
\]
where the last equality follows from the weak law of large numbers. So,
\[
\lim_{n \to \infty} \mathbb{P}(Z_1 + Z_2 + \cdots + Z_n < n(\mu - 1)) = 0.
\]

Proof of (b).
\[
\lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1 + Z_2 + \cdots + Z_n}{n} < \mu\right) = \lim_{n \to \infty} \mathbb{P}(Z_1 + Z_2 + \cdots + Z_n < n\mu)
= \lim_{n \to \infty} \mathbb{P}(Z_1 + Z_2 + \cdots + Z_n - n\mu < 0)
= \lim_{n \to \infty} \mathbb{P}\left(\frac{Z_1 + Z_2 + \cdots + Z_n - n\mu}{\sqrt{n}\sigma} < 0\right)
= \Phi(0) = \frac{1}{2},
\]
where the last equality follows from the central limit theorem.