REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT.

IF YOU DO NOT ASSIGN THE PAGES OF YOUR WORK TO THE QUESTIONS OF THE EXAM IN YOUR UPLOAD TO GRADESCOPE, YOU WILL LOSE POINTS (1 POINT FOR EVERY QUESTION THAT YOU FAIL TO ASSIGN THE PAGES TO).

WRITE YOUR NAME AND STUDENT ID ON THE FIRST PAGE OF YOUR UPLOAD

YOU DO NOT NEED TO COPY DOWN THE QUESTIONS. WRITE YOUR SOLUTIONS TO EACH PROBLEM ON SEPARATE PAGES. CLEARLY INDICATE AT THE TOP OF EACH PAGE THE NUMBER OF THE CORRESPONDING PROBLEM. DIFFERENT PARTS OF THE SAME PROBLEM CAN BE WRITTEN ON THE SAME PAGE (FOR EXAMPLE, PART (A) AND PART (B)).

FROM THE MOMENT YOU ACCESS THE MIDTERM ON GRADESCOPE, YOU WILL HAVE A TOTAL OF 65 MINUTES TO COMPLETE AND UPLOAD YOUR SOLUTIONS TO GRADESCOPE. THE EXAM IS WRITTEN TO TAKE 50 MINUTES. IT IS YOUR RESPONSIBILITY TO UPLOAD YOUR SOLUTIONS ON TIME.
THE MIDTERM IS OPEN TEXTBOOK, AND YOU CAN REWATCH LECTURES AND REVIEW THE LECTURE SLIDES. YOU ARE NOT ALLOWED TO USE CALCULATORS, COMPUTER ALGEBRA SYSTEMS, OR SIMILAR SOFTWARE. YOU ARE NOT ALLOWED TO CONSULT OTHER PEOPLE OR RESOURCES ON THE INTERNET. IN PARTICULAR, CONTRACT CHEATING AND OTHER INSTANCES OF ACADEMIC DISHONESTY CAN RESULT IN SUSPENSION OR DISMISSAL.

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EXCEL WITH INTEGRITY PLEDGE

I pledge to be fair to my classmates and instructors by completing all of my academic work with integrity. This means that I will respect the standards set by the instructor and institution, be responsible for the consequences of my choices, honestly represent my knowledge and abilities, and be a community member that others can trust to do the right thing even when no one is watching. I will always put learning before grades, and integrity before performance. I pledge to excel with integrity.

In addition to the above, I pledge that I did not receive outside assistance with this exam. Outside assistance includes but is not limited to other people, the internet, and resources beyond the textbook, lecture notes/videos, and homework assignments.

To acknowledge that you agree to this pledge, you must copy the sentence:

I choose to excel with integrity as a member of the University of California, San Diego.

Make sure to sign your name below it and date it as well. Exams without this pledge will not be graded.
1. (25 points) An urn contains 5 red balls and 5 green balls. Your friend flips a fair coin: if the coin lands on heads, she puts an extra red ball in the urn; if the coin lands on tails, she puts an extra green ball in the urn. You then draw a ball from the urn.

(a) (15 points) What is the probability that your ball is red?

Answer. Let $R, G, H$, and $T$ stand for the obvious events. Then

\[
P(R) = P(R \cap H) + P(R \cap T) \\
= P(H)P(R|H) + P(T)P(R|T) \\
= \frac{1}{2}\frac{6}{11} + \frac{1}{2}\frac{5}{11} = \frac{1}{2}.
\]

(b) (10 points) Suppose that your ball is red. What is the probability that your friend’s coin landed on heads?

Answer. We can avoid using Bayes’ rule since we have already computed $P(R)$ in part (a) (we also computed $P(R \cap H)$). So,

\[
P(H|R) = \frac{P(H \cap R)}{P(R)} = \frac{\frac{1}{2}\frac{6}{11}}{\frac{1}{2}\frac{6}{11} + \frac{1}{2}\frac{5}{11}} = \frac{6}{11}.
\]
2. (25 points) Let $X$ be uniform on the interval $[0, 10]$. Define $Y = \max(X, 5)$.

(a) (20 points) Determine the CDF of $Y$.

\textit{Answer.} By definition, 

$$Y = \begin{cases} 5 & \text{if } X \leq 5; \\ X & \text{if } X > 5. \end{cases}$$

In particular, $P(Y \leq t) = 0$ for $t < 5$. For $t = 5$,

$$P(Y \leq t) = P(Y \leq 5) = P(Y = 5) = P(X \leq 5) = \frac{5}{10}.$$ 

For $t \in (5, 10]$,

$$P(Y \leq t) = P(X \leq t) = \frac{t}{10}.$$ 

So, we conclude that

$$F_Y(t) = \begin{cases} 0 & \text{if } t < 5; \\ \frac{t}{10} & \text{if } t \in [5, 10]; \\ 1 & \text{if } t \geq 10. \end{cases}$$

\hfill \Box

(b) (5 points) Determine if $Y$ is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of $Y$. If discrete, determine the probability mass function of $Y$. If neither, explain why.

\textit{Answer.} $Y$ is not continuous since $F_Y(t)$ is not continuous (there is a discontinuity at $t = 5$). $Y$ is not discrete since its CDF is not piecewise constant (look at the values $t \in [5, 10]$). \hfill \Box
3. (25 points) Suppose that we have $k$ distinct balls (you can think that they are numbered 1 through $k$) and $n$ distinct boxes (you can think that they are numbered 1 through $n$).

(a) (10 points) You are asked to put away the balls by putting them into the boxes. How many different ways are there to do this? (Note: each box is sufficiently large to hold all $k$ balls by itself if necessary. We do not differentiate between arrangements inside the same box. For example, ball 1 and ball 2 in box 1 is the same as ball 2 and ball 1 in box 1. However, ball 1 and ball 2 in box 1 is different from ball 1 and ball 2 in box 2. Ball 1 and ball 2 in box 1 is also different from ball 3 and ball 4 in box 1.)

Answer. You can think of having $n$ choices of each of the $k$ balls. So, there are $n^k$ possible ways to do this.

(b) (15 points) Suppose that you put the balls away uniformly at random, each of the possible arrangements in part (a) being equally likely. What is the probability that the first box contains exactly 5 balls? (Note: assume that $k \geq 5$).

Answer. We need to choose 5 of the $k$ balls to put in the first box. There are $\binom{k}{5}$ such choices. The remaining $k - 5$ balls can be put into any of the $n - 1$ boxes (we don’t want any more in box 1). So, the answer is

$$\frac{\binom{k}{5}(n - 1)^{k-5}}{n^k}.$$
4. (25 points) Suppose we have a square in $\mathbb{R}^2$ with vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Let $(X,Y)$ denote a uniformly chosen random point inside this square.

(a) (20 points) Let $Z = |X| + |Y|$. Find the CDF of $Z$. (Hint: draw a picture and think about what the absolute value does.)

Answer. The shape looks like a diamond:

$$F_Z(t) = \begin{cases} 
0 & \text{if } t < 0; \\
\frac{2t^2}{4} - \frac{t^2}{2} & \text{if } t \in [0,1]; \\
\frac{4-2(2-t)^2}{4} & \text{if } t \in (1,2]; \\
1 & \text{if } t > 1.
\end{cases}$$
(b) (5 points) Determine if $Z$ is a continuous random variable, a discrete random variable, or neither. If continuous, determine the probability density function of $Z$. If discrete, determine the probability mass function of $Z$. If neither, explain why.

*Answer.* $Z$ is a continuous random variable because its CDF is a continuous function. We can find its PDF by taking the derivative (note that there are finitely many points where the CDF is not differentiable, but this is fine).

$$f_Z(t) = \begin{cases} 0 & \text{if } t < 0; \\ t & \text{if } t \in (0, 1); \\ 2 - t & \text{if } t \in (1, 2); \\ 0 & \text{if } t > 2. \end{cases}$$