

Review

The following topics were covered in this course.

1. **Linear ODE** associated to a symmetric matrix, A , with $\{v_i\}_{i=1}^N$ being an orthogonal basis of eigenvectors of A with $Av_i = \lambda_i v_i$.

a) If $f \in \mathbb{R}^N$ we have

$$f = \sum_{i=1}^N \left((f, |v_i|^{-1} v_i) |v_i|^{-1} v_i \right) = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} v_i.$$

b) The first order linear equation

$$\dot{u}(t) = Au(t) + h(t) \text{ with } u(0) = f$$

has solution

$$u(t) = e^{tA} f + \int_0^t e^{(t-\tau)A} h(\tau) d\tau.$$

where

$$e^{tA} f = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} e^{tA} v_i = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} e^{t\lambda_i} v_i.$$

c) The second order ordinary differential equation,

$$\begin{aligned} \ddot{u}(t) &= Au(t) + h(t) \text{ with} \\ u(0) &= f \text{ and } \dot{u}(0) = g \end{aligned}$$

has a unique solution given by

$$u(t) = \left(\cos \sqrt{-A} t \right) f + \frac{\sin \sqrt{-A} t}{\sqrt{-A}} g + \int_0^t \frac{\sin \sqrt{-A} (t - \tau)}{\sqrt{-A}} h(\tau) d\tau$$

where

$$\cos \sqrt{-A} t f = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} \cos \sqrt{-A} t v_i = \sum_{i=1}^N \frac{(f, v_i)}{\|v_i\|^2} \left(\cos \sqrt{-\lambda_i} t \right) v_i$$

and

$$\frac{\sin \sqrt{-A} t}{\sqrt{-A}} g = \sum_{i=1}^N \frac{(g, v_i)}{\|v_i\|^2} \frac{\sin \sqrt{-A} t}{\sqrt{-A}} v_i = \sum_{i=1}^N \frac{(g, v_i)}{\|v_i\|^2} \left(\frac{\sin \sqrt{-\lambda_i} t}{\sqrt{-\lambda_i}} \right) v_i$$

See Notation 3.14.

2. **Generalities about inner products** on function spaces, see Chapter 5.1.
3. **PDE Examples**

a) The Wave Equation

$$\begin{aligned} u_{tt}(t, x) &= \frac{H}{\delta(x)} u_{xx}(t, x) - g - \frac{k(x)}{\delta(x)} u_t(t, x) \text{ and} \\ u_{tt}(t, x) &= a^2 \Delta u(t, x) + h(t, x) \end{aligned}$$

b) d'Alembert's solution to

$$\begin{aligned} u_{tt} &= a^2 u_{xx} \text{ with} \\ u(0, x) &= f(x) \text{ and } u_t(0, x) = g(x) \end{aligned}$$

is given by

$$u(t, x) = \frac{1}{2} [f(x+at) + f(x-at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(s) ds.$$

c) Heat Equation

$$u_t(t, x) = Lu(t, x) + h(t, x)$$

where

$$Lf(x) := \frac{1}{p(x)} \frac{d}{dx} \left(\kappa(x) \frac{d}{dx} f(x) \right) + \frac{1}{p(x)} q(x) f(x).$$

Such equation come from looking at the heat equation

$$u_t = \Delta u + h \text{ on some domain in } \mathbb{R}^N$$

in spherical and cylindrical coordinates.

d) Equilibrium solutions.

4. (**Strurm-Liouville Spectral Theorem.**) Let L be as above, assume $\kappa > 0$ on $[a, b]$ and let $D = D_B$ or $D = D_{per}$ (in which case we assume additionally that $\kappa(b) = \kappa(a)$), then there exists $u_n \in D$ and $\lambda_n \in \mathbb{R}$ such that:

- a) $-Lu_n = \lambda_n u_n$ for all n ,
- b) the eigenvalues are increasing, i.e.

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots,$$

- c) $\lim_{n \rightarrow \infty} \lambda_n = \infty$ (in fact $\#\{n : \lambda_n \leq a\} \sim a^{1/2}$ or equivalently $\lambda_n \sim n^2$).
- d) Every “nice” function f on $[a, b]$ may be expanded as

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} (f, u_n) u_n(x) \\ &= \sum_{n=1}^{\infty} \left[\int_a^b f(x) u_n(x) p(x) dx \right] u_n(x). \end{aligned}$$

5. Fourier Series

- a) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a 2π -periodic function which is piecewise continuous on $(-\pi, \pi)$. Then at points $x \in \mathbb{R}$ where $f'(x \pm)$ exist we have

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] = \frac{f(x+) + f(x-)}{2}$$

where

$$a_n := (f, \cos n(\cdot)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy \text{ for } n = 0, 1, 2, \dots$$

and

$$b_n := (f, \sin n(\cdot)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy \text{ for } n = 1, 2, \dots$$

Moreover we have

$$\begin{aligned} (f, f) &= \int_{-\pi}^{\pi} |f(x)|^2 dx \\ &= \left(\frac{1}{2}a_0\right)^2 2\pi + \sum_{n=1}^{\infty} \left[a_n^2 \int_{-\pi}^{\pi} \cos^2 nxdx + b_n^2 \int_{-\pi}^{\pi} \sin^2 nxdx \right] \\ &= \frac{\pi}{2}a_0^2 + \pi \sum_{n=1}^{\infty} [a_n^2 + b_n^2]. \end{aligned}$$

This comes from the Sturm-Liouville problems with $L = \frac{d^2}{dx^2}$ with periodic boundary conditions.

- b) (**Cosine Expansion**) Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a piecewise continuous function, then at points $x \in [0, \pi]$ where $f'(x \pm)$ exist we have

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n := \frac{2}{\pi} \int_0^{\pi} f(y) \cos ny \, dy$$

and

$$\begin{aligned} (f, f) &= \int_0^{\pi} |f(x)|^2 dx = \left(\frac{1}{2}a_0\right)^2 \pi + \sum_{n=1}^{\infty} \left[a_n^2 \int_0^{\pi} \cos^2 nxdx \right] \\ &= \frac{\pi}{4}a_0^2 + \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2. \end{aligned}$$

This comes from the Sturm-Liouville problems with $L = \frac{d^2}{dx^2}$ with Neumann boundary conditions.

- c) (**Sine Expansion**) Let $f : [0, \pi] \rightarrow \mathbb{R}$ be a piecewise continuous function, then at points $x \in [0, \pi]$ where $f'(x \pm)$ exist we have

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin nx$$

(except at $x = 0$ and $x = \pi$ where the right side is always zero), where

$$b_n := \frac{2}{\pi} \int_0^{\pi} f(y) \sin ny \, dy.$$

Moreover,

$$(f, f) = \int_0^{\pi} |f(x)|^2 dx = \sum_{n=1}^{\infty} \left[b_n^2 \int_0^{\pi} \sin^2 nxdx \right] = \frac{\pi}{2} \sum_{n=1}^{\infty} a_n^2.$$

This comes from the Sturm-Liouville problems with $L = \frac{d^2}{dx^2}$ with Dirichlet boundary conditions.

6. **Separation of variables techniques** for solving PDE. For example if we want to find the eigenfunctions and eigenvalues to $\Delta u = \lambda u$ with $u = u(x, y)$ for $(x, y) \in [0, \pi]^2$ with Dirichlet Boundary conditions. Then separation of variables would give

$$u_{m,n}(x, y) = \sin mx \cdot \sin ny \tag{C.1}$$

is an orthogonal basis with $\Delta u_{m,n} = -(m^2 + n^2) u_{m,n}$ and

$$\int_0^{\pi} dx \int_0^{\pi} dy u_{m,n}(x, y) u_{m',n'}(x, y) = \delta_{m,m'} \delta_{n,n'} \left(\frac{\pi}{2}\right)^2.$$

Therefore we have the expansion

$$f(x, y) = \sum_{m,n=1}^{\infty} B_{m,n} \sin mx \cdot \sin ny$$

where

$$B_{m,n} = \frac{\int_0^\pi dx \int_0^\pi dy u_{m,n}(x, y) f(x, y)}{\left(\frac{\pi}{2}\right)^2} = \left(\frac{2}{\pi}\right)^2 \int_0^\pi dx \int_0^\pi dy u_{m,n}(x, y) f(x, y).$$

7. **Solving Heat and Wave Equations.** All of the formula for linear ODE hold for heat and wave partial differential equations. For example if $u_{m,n}(x, y)$ are as in Eq. (C.1) then

$$\begin{aligned} u(t, x, y) &= e^{t\Delta} f(x, y) = \sum_{m,n=1}^{\infty} B_{m,n} e^{t\Delta} (\sin mx \cdot \sin ny) \\ &= \sum_{m,n=1}^{\infty} B_{m,n} e^{-t(m^2+n^2)} (\sin mx \cdot \sin ny), \end{aligned}$$

solves the heat equation

$$u_t = \Delta u \text{ with } u(0, x, y) = f(x, y) \text{ and } u = 0 \text{ on the boundary.}$$

More generally if

$$h(t, x, y) = \sum_{m,n=1}^{\infty} H_{m,n}(t) (\sin mx \cdot \sin ny)$$

then

$$\begin{aligned} u(t, x, y) &= e^{t\Delta} f(x, y) + \int_0^t e^{(t-\tau)\Delta} h(\tau, x, y) d\tau \\ &= \sum_{m,n=1}^{\infty} B_{m,n} e^{-t(m^2+n^2)} (\sin mx \cdot \sin ny) \\ &+ \sum_{m,n=1}^{\infty} \int_0^t H_{m,n}(\tau) e^{(t-\tau)\Delta} (\sin mx \cdot \sin ny) d\tau \\ &= \sum_{m,n=1}^{\infty} \left[B_{m,n} e^{-t(m^2+n^2)} + \int_0^t e^{-(t-\tau)(m^2+n^2)} H_{m,n}(\tau) d\tau \right] (\sin mx \cdot \sin ny) \end{aligned}$$

solves the heat equation

$$u_t(t, x, y) = \Delta u(t, x, y) + h(t, x, y) \text{ with } u(0, x, y) = f(x, y) \text{ and } u = 0 \text{ on the boundary.}$$

Similar statements hold for the wave equation,

$$u_t(t, x, y) = \Delta u(t, x, y) + h(t, x, y) \text{ with } u = 0 \text{ on the boundary and } u(0, x, y) = f(x, y) \text{ and } u_t(0, x, y) = g(x, y).$$

Namely

$$\begin{aligned} u(t, x, y) &= \cos\left(t\sqrt{-\Delta}\right) f(x, y) + \frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} g(x, y) \\ &+ \int_0^t \frac{\sin\left((t-\tau)\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} h(\tau, x, y) d\tau \end{aligned}$$

where

$$\begin{aligned} \cos\left(t\sqrt{-\Delta}\right) f(x, y) &= \sum_{m,n=1}^{\infty} B_{m,n} \cos\left(t\sqrt{-(m^2+n^2)}\right) (\sin mx \cdot \sin ny) \\ &= \sum_{m,n=1}^{\infty} B_{m,n} \cos\left(t\sqrt{m^2+n^2}\right) (\sin mx \cdot \sin ny) \end{aligned}$$

and

$$\frac{\sin\left(t\sqrt{-\Delta}\right)}{\sqrt{-\Delta}} f(x, y) = \sum_{m,n=1}^{\infty} B_{m,n} \frac{\sin\left(t\sqrt{m^2+n^2}\right)}{\sqrt{m^2+n^2}} (\sin mx \cdot \sin ny).$$

8. Laplacian in other coordinate systems.

a) Laplacian in Cylindrical Coordinates is

$$\Delta f = \frac{1}{\rho} \partial_\rho (\rho \partial_\rho f) + \frac{1}{\rho^2} \partial_\theta^2 f + \partial_z^2 f.$$

This form of the Laplacian gives rise to Bessel's equation and Bessel functions.

b) Laplacian in Spherical coordinates

$$\begin{aligned} \Delta f &= \frac{1}{r^2} \partial_r (r^2 \partial_r f) + \frac{1}{r^2 \sin \varphi} \partial_\varphi (\sin \varphi \partial_\varphi f) + \frac{1}{r^2 \sin^2 \varphi} \partial_\theta^2 f \\ &= \frac{1}{r} \partial_r^2 (r f) + \frac{1}{r^2 \sin \varphi} \partial_\varphi (\sin \varphi \partial_\varphi f) + \frac{1}{r^2 \sin^2 \varphi} \partial_\theta^2 f. \end{aligned}$$

This form of the Laplacian gives rise to "Legendre polynomials" and more generally "spherical harmonics."

c) You are responsible for problems in these coordinates which do **not** involve the above mentioned special functions (i.e. Bessel functions and Legendre polynomials.)