Study Guide for Math 120A Final (What you should know)

1. \( \mathbb{C} := \{ z = x + iy : x, y \in \mathbb{R} \} \) with \( i^2 = -1 \) and \( \bar{z} = x - iy \). The complex numbers behave much like the real numbers. In particular the quadratic formula holds.

2. \[ |z| = \sqrt{x^2 + y^2} = \sqrt{z \bar{z}}, \quad |zw| = |z||w|, \quad |z + w| \leq |z| + |w|, \quad \text{Re} \, z = \frac{z + \bar{z}}{2}, \quad \text{Im} \, z = \frac{z - \bar{z}}{2i}, \quad \text{Re} \, \bar{z} \leq |z| \quad \text{and} \quad |\text{Im} \, z| \leq |z|. \]

3. \( \{ z : |z - z_0| = \rho \} \) is a circle of radius \( \rho \) centered at \( z_0 \).

4. \( e^z = e^x \cos y + i \sin y \), \quad |e^z| = e^x \leq e^{|z|} \quad \text{and} \quad z = |z|e^{i\theta} \) for some \( \theta \in \mathbb{R} \) for every \( z \in \mathbb{C} \).

5. \( \arg(z) = \{ \theta \in \mathbb{R} : z = |z|e^{i\theta} \} \) and \( \text{Arg}(z) = \theta \) if \(-\pi < \theta \leq \pi \quad \text{and} \quad z = |z|e^{i\theta} \).

6. \( z^{1/n} = \sqrt[n]{|z|}e^{\frac{i\arg(z)}{n}} \).

7. More generally if \( c \in \mathbb{C} \) we set \( z^c := e^{c \log(z)} \) and if \( \ell \) is a branch of \( \log \), the we define \( z^\ell := e^{\ell(z)} \) to be a branch of \( z^c \). With this notation we have

\[ \frac{d}{dz} z^\ell = cz^{\ell-1}. \]

8. \( \lim_{z \to z_0} f(z) = L \). Usual limit rules hold from real variables.


10. The definition of complex differentiable \( f(z) \). Examples, \( p(z), e^z, e^{p(z)}, 1/z, 1/p(z) \) etc.

11. Key points of \( e^z \) are \( \frac{d}{dz} e^z = e^z \) and \( e^z e^w = e^{z+w} \).

12. All of the usual derivative formulas hold, in particular product, sum, and chain rules:

\[ \frac{d}{dz} f(g(z)) = f'(g(z)) g'(z) \]

and

\[ \frac{d}{dt} f(z(t)) = f'(z(t)) \dot{z}(t). \]

13. \( \text{Re} \, z, \text{Im} \, z, \bar{z} \), are nice functions from the real - variables point of view but are not complex differentiable.

14. Integration:

\[ \int_a^b z(t) \, dt := \int_a^b x(t) \, dt + i \int_a^b y(t) \, dt. \]

All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.

15. You should know; if \( f \) is complex differentiable at \( z_0 \), then Cauchy Riemann (C.R.) equations hold at a point \( z_0 \in \mathbb{C} \), i.e.

\[ f_y = i f_x \quad \text{or equivalently if} f = u + iv \quad \text{then} u_y = -v_x \quad \text{and} \quad u_x = v_y \quad \text{at} \ z_0. \]

Conversely, if the C.R. equations hold and the partial derivatives are continuous near some point \( z \) then \( f'(z) \) exists and \( f'(z) = f_x(z) = -if_y(z) \).

16. You should understand and be able to use the following analytic functions:

- a) \( e^z = e^x \cos y + i \sin y = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \).
- b) \( \frac{1}{z} = \sum_{n=0}^{\infty} z^n \) for \( |z| < 1 \).
- c) \( \log z = \ln |z| + i \arg \, z \) and its branches, \( \ell(z) \). That is \( \ell \) is continuous and satisfies \( e^{\ell(z)} = z \) for \( z \) in the domain of \( \ell \). No matter the branch we have \( \frac{d}{dz} \ell(z) = \frac{1}{z} \).
- d) If Log is the principal branch of Log we have seen,

\[ \text{Log} \, (1 - z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} \quad \text{if} |z| < 1. \]

- e) \( \sin(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \).
- f) \( \cos(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \).
- g) \( \sinh(z) := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \).
- h) \( \cosh(z) := \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \).
- i) \( \tan(z) := -i \sum_{n=0}^{\infty} \frac{(-1)^n \frac{z^{2n+1}}{(2n+1)!} \frac{1}{2^{2n+1}}} \).
- j) \( \tanh(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{e^{2n+1} - e^{-2n+1}} \).

17. Be able to parametrize simple contours.

18. Be able to compute contour integrals by parametrizing the contour to get

\[ \int_C f(z) \, dz = \int_a^b f(z(t)) \dot{z}(t) \, dt. \]
19. Be able to estimate contour integrals using
\[ \left| \int_C f(z) \, dz \right| \leq \max_{z \in C} |f(z)| \cdot \text{length}(C). \]

20. Be able to compute contour integrals using the fundamental theorem of calculus: if \( f \) is analytic on a neighborhood of a contour \( C \), then
\[ \int_C f'(z) \, dz = f(C_{\text{end}}) - f(C_{\text{begin}}). \]

21. Be able to use the Cauchy-Goursat theorem to argue that \( \int_{C_1} f(z) \, dz = \int_{C_2} f(z) \, dz \) when \( C_1 \) and \( C_2 \) are appropriately homotopic in the domain of definition of the analytic function, \( f \).

22. Be able to compute residues of \( h \) at \( z_0 \),
\[ \text{res}_{z=z_0} h(z) := \lim_{\rho \to 0} \frac{1}{2\pi i} \oint_{|z-z_0| = \rho} h(z) \, dz, \]
and use the residue theorem for computing contour integrals. The basic methods we have learned for computing residues are (assuming \( f \) and \( g \) are analytic near \( z_0 \));
   a) If \( h(z) = \frac{f(z)}{z-z_0} \), then \( \text{res}_{z=z_0} h(z) = \text{res}_{z=z_0} \frac{f(z)}{z-z_0} = f(z_0) \) or more generally,
   b) If \( h(z) = \frac{f(z)}{(z-z_0)^{n+1}} \), then \( \text{res}_{z=z_0} h(z) = \text{res}_{z=z_0} \frac{f(z)}{(z-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0) \),
   c) If \( h(z) = \frac{f(z)}{g(z)} \), then \( \text{res}_{z=z_0} h(z) = \text{res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)} \) provided \( g(z_0) = 0 \) and \( g'(z_0) \neq 0 \). **Warning:** this formula is not valid if \( g(z_0) \neq 0 \) or if \( g'(z_0) = 0 \).
   d) If \( h(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \) then \( \text{res}_{z=z_0} h(z) = a_{-1} \).

23. Be able to use complex techniques to compute real integrals similar to those that have appeared in the homework problems and/or in class.

24. Be able to compute Taylor series and Laurent series expansions (in simple cases) of a function \( f \) centered at a point \( z_0 \in \mathbb{C} \). **Hint:** If \( z_0 \neq 0 \), write \( z = z_0 + h \) and then do the expansion in \( h \) about \( h = 0 \). At the end replace \( h \) by \( z - z_0 \).