Study Guide for Math 120A Final (What you should know)

1. $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$ with $i^2 = -1$ and $\bar{z} = x - iy$. The complex numbers behave much like the real numbers. In particular the quadratic formula holds.
2. $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$, $|zw| = |z|\,|w|$, $|z + w| \le |z| + |w|$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$, $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$. We also have $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z + w} = \bar{z} + \bar{w}$ and $z^{-1} = \frac{\bar{z}}{|z|^2}$.
3. $\{z : |z - z_0| = \rho\}$ is a circle of radius ρ centered at z_0.
   $\{z : |z - z_0| < \rho\}$ is the open disk of radius ρ centered at z_0.
   $\{z : |z - z_0| \ge \rho\}$ is every thing outside of the open disk of radius ρ centered at z_0.
4. $e^z = e^x (\cos y + i \sin y)$, $|e^z| = e^x = e^{\operatorname{Re} z} \le e^{|z|}$ and $z = |z|\, e^{i\theta}$ for some $\theta \in \mathbb{R}$ for every $z \in \mathbb{C}$.
5. $\arg(z) = \left\{\theta \in \mathbb{R} : z = |z|\, e^{i\theta}\right\}$ and $\operatorname{Arg}(z) = \theta$ if $-\pi < \theta \le \pi$ and $z = |z|\, e^{i\theta}$. Notice that $z = |z|\, e^{i \arg(z)}$
6. $z^{1/n} = \sqrt[n]{|z|} e^{i \frac{\arg(z)}{n}}$.
7. More generally if $c \in \mathbb{C}$ we set $z^c := e^{c \log(z)}$ and if ℓ is a branch of log, the we define $z_\ell^c := e^{c\ell(z)}$ to be a branch of z^c. With this notation we have
$$\frac{d}{dz} z_\ell^c = c z_\ell^{c-1}.$$
8. $\lim_{z \to z_0} f(z) = L$. Usual limit rules hold from real variables.
9. Mapping properties of simple complex functions.
10. The definition of complex differentiable $f(z)$. Examples, $p(z)$, e^z, $e^{p(z)}$, $1/z$, $1/p(z)$ etc.
11. Key points of e^z are is $\frac{d}{dz} e^z = e^z$ and $e^z e^w = e^{z+w}$.
12. All of the usual derivative formulas hold, in particular product, sum, and chain rules:
$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$
and
$$\frac{d}{dt} f(z(t)) = f'(z(t))\, \dot{z}(t).$$
13. $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, are nice functions from the real - variables point of view but are **not** complex differentiable.
14. Integration:
$$\int_a^b z(t)\, dt := \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt.$$
All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.
15. You should know; if f is complex differentiable at z_0, then Cauchy Riemann (C.R.) equations hold at a point $z_0 \in \mathbb{C}$, i.e.
$$f_y = i f_x \text{ or equivalently if } f = u + iv \text{ then } u_y = -v_x \text{ and } u_x = v_y \text{ at } z_0.$$
Conversely, if the C.R. equations hold and the partial derivatives are continuous near some point z then $f'(z)$ exists and $f'(z) = f_x(z) = -i f_y(z)$.
16. You should understand and be able to use the following analytic functions:
    a) $e^z = e^x (\cos y + i \sin y) = \sum_{n=0}^\infty \frac{1}{n!} z^n$.
    b) $\frac{1}{1-z} = \sum_{n=0}^\infty z^n$ for $|z| < 1$.
    c) $\log z = \ln |z| + i \arg z$ and its branches, $\ell(z)$. That is ℓ is continuous and satisfies $e^{\ell(z)} = z$ for z in the domain of ℓ. No matter the branch we have $\frac{d}{dz} \ell(z) = \frac{1}{z}$.
    d) If Log is the principal branch of Log we have seen,
$$\operatorname{Log}(1 - z) = -\sum_{n=0}^\infty \frac{1}{n+1} z^{n+1} \text{ if } |z| < 1.$$
    e) $\sin(z) := \frac{e^{iz} - e^{-iz}}{2i} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} z^{2n+1}$.
    f) $\cos(z) := \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} z^{2n}$.
    g) $\sinh(z) := \frac{e^z - e^{-z}}{2} = \sum_{n=0}^\infty \frac{1}{(2n+1)!} z^{2n+1}$.
    h) $\cosh(z) := \frac{e^z + e^{-z}}{2} = \sum_{n=0}^\infty \frac{1}{(2n)!} z^{2n}$.
    i) $\tan(z) = \frac{\sin(z)}{\cos(z)} = -i \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$.
    j) $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$.
17. Be able to parametrize simple contours.
18. Be able to compute contour integrals by parametrizing the contour to get
$$\int_C f(z)\, dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt.$$

19. Be able to estimate contour integrals using

$$\left|\int_C f(z)\,dz\right| \le \max_{z\in C}|f(z)|\cdot \text{length}(C).$$

20. Be able to compute contour integrals using the fundamental theorem of calculus: if f is analytic on a neighborhood of a contour C, then

$$\int_C f'(z)\,dz = f(C_{\text{end}}) - f(C_{\text{begin}}).$$

21. Be able to use the Cauchy-Goursat theorem to argue that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ when C_1 and C_2 are appropriately homotopic in the domain of definition of the analytic function, f.
22. Be able to compute residues of h at z_0,

$$\text{res}_{z=z_0} h(z) := \lim_{\rho\downarrow 0}\frac{1}{2\pi i}\oint_{|z-z_0|=\rho} h(z)\,dz,$$

and use the residue theorem for computing contour integrals. The basic methods we have learned for computing residues are (assuming f and g are analytic near z_0);

a) If $h(z) = \frac{f(z)}{z-z_0}$, then $\text{res}_{z=z_0} h(z) = \text{res}_{z=z_0}\frac{f(z)}{z-z_0} = f(z_0)$ or more generally,
b) If $h(z) = \frac{f(z)}{(z-z_0)^{n+1}}$, then $\text{res}_{z=z_0} h(z) = \text{res}_{z=z_0}\frac{f(z)}{(z-z_0)^{n+1}} = \frac{1}{n!}f^{(n)}(z_0)$,
c) If $h(z) = \frac{f(z)}{g(z)}$, then $\text{res}_{z=z_0} h(z) = \text{res}_{z=z_0}\frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$ **provided** $g(z_0) = 0$ and $g'(z_0) \ne 0$. **Warning:** this formula is **not valid** if $g(z_0) \ne 0$ or if $g'(z_0) = 0$.
d) If $h(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ then $\text{res}_{z=z_0} h(z) = a_{-1}$.

23. Be able to use complex techniques to compute real integrals similar to those that have appeared in the homework problems and/or in class.
24. Be able to compute Taylor series and Laurent series expansions (in **simple cases)** of a function f centered at a point $z_0 \in \mathbb{C}$. **Hint:** If $z_0 \ne 0$, write $z = z_0 + h$ and then do the expansion in h about $h = 0$. At the end replace h by $z - z_0$.