Study Guide for Math 120A Midterm 1 (What you should know)

1. $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$ with $i^2 = -1$ and $\bar{z} = x - iy$. The complex numbers behave much like the real numbers. In particular the quadratic formula holds.
2. $|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$, $|zw| = |z|\,|w|$, $|z + w| \le |z| + |w|$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$, $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.
3. We also have $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z+w} = \bar{z} + \bar{w}$ and $z^{-1} = \frac{\bar{z}}{|z|^2}$.
4. $\{z : |z - z_0| = \rho\}$ is a circle of radius ρ centered at z_0.
   $\{z : |z - z_0| < \rho\}$ is the open disk of radius ρ centered at z_0.
   $\{z : |z - z_0| \ge \rho\}$ is every thing outside of the open disk of radius ρ centered at z_0.
5. $e^z = e^x (\cos y + i \sin y)$, every $z = |z|\, e^{i\theta}$.
6. Know that $e^z e^w = e^{z+w}$ for all $z, w \in \mathbb{C}$.
7. $|e^z| = e^x = e^{\operatorname{Re} z} \le e^{|z|}$.
8. $\arg(z) = \left\{\theta \in \mathbb{R} : z = |z|\, e^{i\theta}\right\}$ and $\operatorname{Arg}(z) = \theta$ if $-\pi < \theta \le \pi$ and $z = |z|\, e^{i\theta}$. Notice that $z = |z|\, e^{i \arg(z)}$
9. $z^{1/n} = \sqrt[n]{|z|} e^{i\frac{\arg(z)}{n}} = \sqrt[n]{|z|} \left\{e^{i\frac{\operatorname{Arg}(z)+k2\pi}{n}} : k = 0, 1, 2, \dots, n-1\right\}$.
10. Differentiation of complex functions, $z(t) = a(t) + ib(t)$, of one real variable t. Recall that
$$\dot{z}(t) = \dot{a}(t) + i\dot{b}(t)$$
and that the usual rules hold like
$$\frac{d}{dt}\left[z(t) + w(t)\right] = \dot{z}(t) + \dot{w}(t)$$
and
$$\frac{d}{dt}\left[z(t)\, w(t)\right] = \dot{z}(t)\, w(t) + z(t)\, \dot{w}(t).$$
11. Integration:
$$\int_a^b z(t)\, dt := \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt.$$
All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.
12. Know how to estimate the size of integrals using,
$$\left|\int_a^b z(t)\, dt\right| \le \int_a^b |z(t)|\, dt \le M \cdot (b - a)$$
where $M \ge 0$ is any constant such that
$$|z(t)| \le M \text{ for } a \le t \le b.$$
13. Know how to compute simple limits $\lim_{z \to z_0} f(z) = L$ where $z_0, L \in \mathbb{C} \cup \{\infty\}$. (Usual limit rules hold from real variables.)
14. Basic mapping properties of simple complex functions.
15. The definition of complex differentiability. Examples, $p(z)$, e^z, $e^{p(z)}$, $1/z$, $1/p(z)$ etc.
16. All of the usual derivative formulas hold, in particular product, sum, and chain rules:
$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$
and
$$\frac{d}{dt} f(z(t)) = f'(z(t))\, \dot{z}(t).$$
where f, g are analytic and $z(t) \in \mathbb{C}$ is continuously differentiable.
17. $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, are nice functions from the real - variables point of view but are **not** complex differentiable.
18. The Cauchy Riemann (C.R.) equations hold,
$$f_y = i f_x \text{ or equivalently if } f = u + iv \text{ then } u_y = -v_x \text{ and } u_x = v_y$$
if f is complex differentiable.
19. Conversely, if the C.R. equations hold and the partial derivatives are continuous near some point z then $f'(z)$ exists and $f'(z) = f_x(z) = -i f_y(z)$.