# 29

## (12/3/2018) More Laurent Series

**Lemma 29.1.** *For $m, n \in \mathbb{Z}$ we have*

$$\frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{(z-z_0)^n}{(z-z_0)^{m+1}}dz = \delta_{m,n} = \begin{cases} 1 \text{ if } m=n \\ 0 \text{ if } m \neq n \end{cases}.$$

*Remark 29.2 (Uniqueness of the Laurent coeffients).* If we know somehow that

$$f(z) = \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n \text{ for } r < |z-z_0| < R,$$

then for $r < \rho < R$,

$$\begin{aligned} a_m &:= \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(z)}{(z-z_0)^{m+1}}dz \\ &= \frac{1}{2\pi i}\oint_{C_\rho} \frac{1}{(z-z_0)^{m+1}} \sum_{n=-\infty}^{\infty} b_n (z-z_0)^n dz \\ &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} b_n \oint_{C_\rho} \frac{1}{(z-z_0)^{m+1}}(z-z_0)^n dz \\ &= \sum_{n=-\infty}^{\infty} b_n \delta_{mn} = b_m. \end{aligned}$$

So the coefficients in the Laurent expansion are unique.

**Definition 29.3 ($O-$ notation).** *We write $f(z) = O(z^N)$ for any $N \in \mathbb{Z}$ to mean, there is a constant $C < \infty$ such that $|f(z)| \le C|z|^N$ for $z \neq 0$ but near zero.*

**Proposition 29.4 (Manipulating Series).** *Suppose $f(z)$ is analytic on $D'(z_0,\varepsilon) := D(z_0,\varepsilon) \setminus \{z_0\}$, with*

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z-z_0)^m \text{ where}$$

$$a_m = \frac{1}{2\pi i}\oint_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}dz \text{ with } 0<\rho<\varepsilon.$$

*If we also know for some $K, N \in \mathbb{Z}$ with $K \le N$ that*

$$f(z) = \sum_{n=K}^{N} b_n (z-z_0)^n + O\left((z-z_0)^{N+1}\right). \tag{29.1}$$

*then for $m \le N$,*

$$a_m = b_m 1_{K \le m \le N} = \begin{cases} 0 \text{ for } m<K \\ b_m \text{ for } K \le m \le N \end{cases}.$$

*If $f$ is analytic on $D(z_0,\varepsilon)$ and*

$$f(z) = \sum_{n=0}^{N} b_n (z-z_0)^n + O\left((z-z_0)^{N+1}\right), \tag{29.2}$$

*then*

$$b_n = \frac{f^{(n)}(z_0)}{n!} \text{ for } n \le N. \tag{29.3}$$

**Proof.** As usual we assume that $z_0 = 0$ to simplify notation. We then have, for any $\rho > 0$, that

$$\begin{aligned} a_m &= \frac{1}{2\pi i}\oint_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{m+1}}dz \\ &= \frac{1}{2\pi i}\oint_{|z-z_0|=\rho} \left[\sum_{n=K}^{N} b_n \frac{(z-z_0)^n}{(z-z_0)^{m+1}} + \frac{O\left((z-z_0)^{N+1}\right)}{(z-z_0)^{m+1}}\right] dz \\ &= \sum_{n=K}^{N} b_n \delta_{m,n} + \frac{1}{2\pi i}\oint_{|z-z_0|=\rho} O\left((z-z_0)^{N-m}\right) dz \\ &= b_m 1_{K \le m \le N} + R_m(\rho), \end{aligned}$$

If $m \le N$, then

$$|R_m(\rho)| = \left|\frac{1}{2\pi i}\oint_{|z-z_0|=\rho} O\left((z-z_0)^{N-m}\right) dz\right| \le \frac{1}{|2\pi i|} C 2\pi\rho \to 0 \text{ as } \rho \downarrow 0.$$

and we have shown

$$a_m = b_m 1_{K \le m \le N} \text{ for all } m \le N$$

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We will now use this in conjunction with the already proved result.

**Proposition 29.5.** *Suppose that* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *is a convergent power series expansion near* $z = 0$*, then*

$$f(z) = \sum_{n=0}^{N} a_n z^n + O\left(z^{N+1}\right).$$

## 29.1 Residue Calculations and Uses

**Proposition 29.6.** *If* $f$ *is analytic near* $z_0$ *and* $n \geq 0$*, then*

$$res_{z_0} \frac{f(z)}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}$$

**Proof.** We have

$$\frac{f(z)}{(z-z_0)^{n+1}} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \frac{(z-z_0)^k}{(z-z_0)^{n+1}} = \dots \frac{f^{(n)}(z_0)}{n!} \frac{1}{z-z_0} + \dots.$$

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**Exercise 29.1.** Use the residue theorem to show:

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = \frac{\pi}{2}.$$

**Solution to Exercise (29.1).** The usual techniques (**see class notes for details which will be needed on the final!**) shows,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx &= 2\pi i \cdot \mathrm{Res}_{z=i} \frac{1}{(1+z^2)^2} \\ &= 2\pi i \cdot \mathrm{Res}_{z=i} \left[\frac{1}{(z-i)^2} (z+i)^{-2}\right] \\ &= 2\pi i (-2)(z+i)^{-3}|_{z=i} = \frac{-4\pi i}{8i^3} = \frac{\pi}{2}.\end{aligned}$$

**Corollary 29.7.** *Suppose that* $f$ *and* $g$ *are analytic near* $z_0$*,* $g(z_0) = 0$ *but* $g'(z_0) \neq 0$*, then*

$$\mathrm{res}_{z_0}\left(\frac{f}{g}\right) = \frac{f(z_0)}{g'(z_0)}.$$

**Proof.** ( Assume $z_0 = 0$ for simplicity, did not do this in class!) We have

$$\frac{f(z)}{g(z)} = \frac{f(z)}{g'(0)z + O(z^2)} = \frac{1}{z} h(z) \text{ where } h(z) = \frac{f(z)}{g'(0) + \frac{1}{2}g''(0)z + \dots}$$

and so

$$res_0 \frac{f(z)}{g(z)} = h(0) = \frac{f(0)}{g'(0)}.$$

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**Corollary 29.8 (Highly Optional!).** *Suppose that* $f$ *and* $g$ *are analytic near* $z_0$ *and* $g$ *has a zero of order* 2*, i.e. (i.e.* $g(z_0) = 0 = g'(z_0)$*but* $g''(z_0) \neq 0$*) then*

$$\mathrm{res}_{z_0}\left(\frac{f}{g}\right) = 2\frac{f'(z_0)g''(z_0) - \frac{1}{3}g'''(z_0)f(z_0)}{g''(z_0)^2}.$$

**Proof.** In this case

$$\frac{f(z)}{g(z)} = \frac{f(z)}{\frac{1}{2}g''(0)z^2 + \frac{1}{3!}g'''(0)z^3 + \dots} = \frac{1}{z^2}h(z),$$

where

$$h(z) = \frac{f(z)}{\frac{1}{2}g''(0) + \frac{1}{3!}g'''(0)z + \dots}$$

and hence by Proposition 29.6,

$$\begin{aligned} res_0 \frac{f(z)}{g(z)} = h'(0) &= \frac{f'(0)\frac{1}{2}g''(0) - \frac{1}{3!}g'''(0)f(0)}{\left[\frac{1}{2}g''(0)\right]^2} \\ &= 4\frac{f'(0)\frac{1}{2}g''(0) - \frac{1}{3!}g'''(0)f(0)}{g''(0)^2} = 2\frac{f'(0)g''(0) - \frac{1}{3}g'''(0)f(0)}{g''(0)^2}.\end{aligned}$$

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