6.2 Taylor's Theorem Problems

Theorem 6.8 (Taylor's at order 1 and 2). *If $f(t)$ for $0 \le t \le 1$ is twice continuously differentiable, then*

$$f(1) = f(0) + \int_0^1 \dot{f}(t)\, dt \text{ and}$$

$$f(1) = f(0) + \frac{\dot{f}(0)}{1!} + \frac{1}{1!}\int_0^1 \ddot{f}(t)(1-t)\, dt.$$

Proof. The first assertion follows by the fundamental theorem of calculus

$$f(1) - f(0) = \int_0^1 \dot{f}(t)\, dt.$$

For the second we integrate by parts as follows;

$$\int_0^1 \dot{f}(t)\, dt = -\int_0^1 \dot{f}(t)\, d(1-t) = -\dot{f}(t)(1-t)\,|_0^1 + \int_0^1 \ddot{f}(t)(1-t)\, dt$$

and therefore

$$f(1) = f(0) + \int_0^1 \dot{f}(t)\, dt = f(0) + \frac{\dot{f}(0)}{1!} + \frac{1}{1!}\int_0^1 \ddot{f}(t)(1-t)\, dt.$$

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Exercise 6.1. If $f : \mathbb{R} \to \mathbb{C}$ is a function which is differentiable to all orders, show for all $N \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ that

$$f(1) = \sum_{k=0}^{N} \frac{1}{k!} f^{(k)}(0) + \frac{1}{N!}\int_0^1 f^{(N+1)}(t)(1-t)^N\, dt.$$

Hint: use integration by parts and induction with Theorem 6.8 providing the case $N = 0$ and $N = 1$.

Exercise 6.2. Recall the if we define $e^z := e^x(\cos y + i \sin y)$ where $z = x + iy$, then $\frac{d}{dt}e^{tz} = ze^{tz}$. Use Exercise 6.1 with $f(t) = e^{tz}$ to conclude;

$$e^z = f(1) = \sum_{k=0}^{N} \frac{z^k}{k!} + R_N(z) \tag{6.4}$$

where

$$R_N(z) = \frac{z^{N+1}}{N!}\int_0^1 e^{tz}(1-t)^N\, dt. \tag{6.5}$$

Then show that $\lim_{N\to\infty} |R_N(z)| = 0$ for all $z \in \mathbb{C}$ and use this to conclude

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ for all } z \in \mathbb{C}.$$