Study Guide for Math 120A Exam 2 (What you should know)

1. You may need general knowledge of material from the start of the course. For example you should recall that $z^{1/n} = \sqrt[n]{|z|}e^{i\frac{\arg(z)}{n}}$ where $\arg(z) = \{\theta \in \mathbb{R} : z = |z| e^{i\theta}\}$.
2. Be familiar with the following analytic functions;
   a) $\sin(z) := \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$
   b) $\cos(z) := \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$
   c) $\sinh(z) := \frac{e^{z}-e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}$
   d) $\cosh(z) := \frac{e^{z}+e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$
   e) $\tan(z) = \frac{\sin(z)}{\cos(z)} = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$
   f) $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$.

   [**You do not need the power series formula above for test 2 but will likely be used on the final exam.**]
3. Be familiar with $\log z = \ln|z| + i \arg z$ and its branches. I will denote a typical branch of log by ℓ. Recall that $\ell'(z) = 1/z$ for all branches, ℓ, of log.
4. $z^{1/n} = \sqrt[n]{|z|}e^{i\frac{\arg(z)}{n}}$ – a branch of $z^{1/n}$ is $z_\ell^{1/n} := e^{\frac{1}{n}l(z)}$.
5. More generally if $c \in \mathbb{C}$ we set $z^c := e^{c\log(z)}$ and if ℓ is a branch of log, the we define $z_\ell^c := e^{c\ell(z)}$ to be a branch of z^c. With this notation we have
$$\frac{d}{dz} z_\ell^c = c z_\ell^{c-1}.$$
6. You should have some familiarity with some of the inverse trig. functions as well.
7. You should know all of the usual derivative formulas for complex derivatives hold including the sum, product, and chain rules;
$$\frac{d}{dz} f(g(z)) = f'(g(z))\, g'(z)$$
and
$$\frac{d}{dt} f(z(t)) = f'(z(t))\, \dot{z}(t).$$
where f, g are analytic and $z(t) \in \mathbb{C}$ is continuously differentiable. You should understand how these rules can be used to show a function is analytic on some open set.
8. You should be able to apply the **"converse" to the chain rule** covered in class in order to show continuous solutions, $f(z)$, to equations of the form $G(f(z)) = h(z)$ are analytic when G and h are analytic functions and $G'(f(z)) \neq 0$. Recall in this case that
$$f'(z) = \frac{h'(z)}{G'(f(z))}.$$
9. Integration:
$$\int_a^b z(t)\, dt := \int_a^b x(t)\, dt + i \int_a^b y(t)\, dt.$$
All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.
10. Be able to **parametrize simple contours.**
11. Be able to compute contour integrals by parametrizing the contour and then evaluating the integral using
$$\int_C f(z)\, dz = \int_a^b f(z(t))\, \dot{z}(t)\, dt$$
where C is the contour $z = z(t)$ with $a \leq t \leq b$. [Remember you are formally letting $z = z(t)$!]
12. Be able to compute contour integrals using the **fundamental theorem of calculus**: **if** f is analytic on a neighborhood of a contour C, then
$$\int_C f'(z)\, dz = f(C_{\text{end}}) - f(C_{\text{begin}}).$$
In particular under these assumptions
$$\int_C f'(z)\, dz = 0 \text{ whenever } C \text{ is a closed loop.}$$
13. Be able to estimate contour integrals using
$$\left|\int_C f(z)\, dz\right| \leq M \cdot L \text{ where}$$
$$|f(z)| \leq M \text{ for } z \in C \text{ and } L = \text{length}(C).$$