Study Guide for Math 120A Final (What you should know)

1. $\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}$ with $i^2 = -1$ and $\bar{z} = x - iy$. The complex numbers behave much like the real numbers. In particular the quadratic formula holds.
2. $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$, $|zw| = |z|\,|w|$, $|z+w| \le |z| + |w|$, $\operatorname{Re} z = \frac{z+\bar{z}}{2}$, $\operatorname{Im} z = \frac{z-\bar{z}}{2i}$, $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$. We also have $\overline{zw} = \bar{z}\bar{w}$ and $\overline{z+w} = \bar{z} + \bar{w}$ and $z^{-1} = \frac{\bar{z}}{|z|^2}$.
3. $\{z : |z - z_0| = \rho\}$ is a circle of radius ρ centered at z_0.
   $\{z : |z - z_0| < \rho\}$ is the open disk of radius ρ centered at z_0.
   $\{z : |z - z_0| \ge \rho\}$ is every thing outside of the open disk of radius ρ centered at z_0.
4. $e^z = e^x(\cos y + i \sin y)$, $|e^z| = e^x = e^{\operatorname{Re} z} \le e^{|z|}$ and $z = |z|\,e^{i\theta}$ for some $\theta \in \mathbb{R}$ for every $z \in \mathbb{C}$.
5. $\arg(z) = \left\{\theta \in \mathbb{R} : z = |z|\,e^{i\theta}\right\}$ and $\operatorname{Arg}(z) = \theta$ if $-\pi < \theta \le \pi$ and $z = |z|\,e^{i\theta}$. Notice that $z = |z|\,e^{i\arg(z)}$
6. $z^{1/n} = \sqrt[n]{|z|}e^{i\frac{\arg(z)}{n}}$.
7. More generally if $c \in \mathbb{C}$ we set $z^c := e^{c\log(z)}$ and if ℓ is a branch of log, the we define $z_\ell^c := e^{c\ell(z)}$ to be a branch of z^c. With this notation we have
$$\frac{d}{dz}z_\ell^c = cz_\ell^{c-1}.$$
8. $\lim_{z\to z_0} f(z) = L$. Usual limit rules hold from real variables.
9. Mapping properties of simple complex functions.
10. The definition of complex differentiable $f(z)$. Examples, $p(z)$, e^z, $e^{p(z)}$, $1/z$, $1/p(z)$ etc.
11. Key points of e^z are is $\frac{d}{dz}e^z = e^z$ and $e^z e^w = e^{z+w}$.
12. All of the usual derivative formulas hold, in particular product, sum, and chain rules:
$$\frac{d}{dz}f(g(z)) = f'(g(z))\,g'(z)$$
and
$$\frac{d}{dt}f(z(t)) = f'(z(t))\,\dot{z}(t).$$
13. $\operatorname{Re} z$, $\operatorname{Im} z$, $\bar{z}$, are nice functions from the real - variables point of view but are **not** complex differentiable.
14. Integration:
$$\int_a^b z(t)\,dt := \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt.$$
All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.
15. You should know; if f is complex differentiable at z_0, then Cauchy Riemann (C.R.) equations hold at a point $z_0 \in \mathbb{C}$, i.e.
$$f_y = if_x \text{ or equivalently if } f = u + iv \text{ then } u_y = -v_x \text{ and } u_x = v_y \text{ at } z_0.$$
Conversely, if the C.R. equations hold and the partial derivatives are continuous near some point z then $f'(z)$ exists and $f'(z) = f_x(z) = -if_y(z)$.
16. You should understand and be able to use the following analytic functions:
    a) $e^z = e^x(\cos y + i \sin y) = \sum_{n=0}^\infty \frac{1}{n!}z^n$.
    b) $\log z = \ln|z| + i\arg z$ and its branches:
$$\operatorname{Log}(1-z) = -\sum_{n=0}^\infty \frac{1}{n+1}z^{n+1} \text{ if } |z| < 1.$$
    c) z^a and its branches: if $(1+z)^\alpha = e^{\alpha \operatorname{Log}(1+z)}$ then
$$(1+z)^\alpha = \sum_{n=0}^\infty \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}z^n$$
    in particular if $\alpha = -1$, then
$$\frac{1}{1-z} = \sum_{n=0}^\infty z^n.$$
    d) $\sin(z) := \frac{e^{iz}-e^{-iz}}{2i} \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!}z^{2n+1}$
    e) $\cos(z) := \frac{e^{iz}+e^{-iz}}{2} = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}z^{2n}$
    f) $\sinh(z) := \frac{e^z-e^{-z}}{2} = \sum_{n=0}^\infty \frac{1}{(2n+1)!}z^{2n+1}$
    g) $\cosh(z) := \frac{e^z+e^{-z}}{2} = \sum_{n=0}^\infty \frac{1}{(2n)!}z^{2n}$
    h) $\tan(z) = \frac{\sin(z)}{\cos(z)} = -i\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$

i) $\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$

17. Be able to compute contour integrals by parametrizing the contour to get

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,\dot{z}(t)\,dt.$$

18. Be able to estimate contour integrals using

$$\left|\int_C f(z)\,dz\right| \le \max_{z\in C} |f(z)| \cdot \text{length}(C).$$

19. Be able to compute contour integrals using the fundamental theorem of calculus: if f is analytic on a neighborhood of a contour C, then

$$\int_C f'(z)\,dz = f(C_{\text{end}}) - f(C_{\text{begin}}).$$

20. Be able to use the Cauchy-Goursat theorem to argue that $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ when C_1 and C_2 are appropriately homotopic in the domain of definition of the analytic function, f.
21. Be able to compute residues of f at z_0,

$$\text{res}_{z=z_0} f(z) := \lim_{\rho\downarrow 0} \frac{1}{2\pi i} \oint_{|z-z_0|=\rho} f(z)\,dz,$$

and use the residue theorem for computing contour integrals. The basic methods we have learned for computing residues are (assuming f and g are analytic near z_0);

a) $\text{res}_{z=z_0} \frac{f(z)}{z-z_0} = f(z_0)$ or more generally,

b) $\text{res}_{z=z_0} \frac{f(z)}{(z-z_0)^{n+1}} = \frac{1}{n!} f^{(n)}(z_0)$, and

c) $\text{res}_{z=z_0} \frac{f(z)}{g(z)} = \frac{f(z_0)}{g'(z_0)}$ **provided** $g(z_0) = 0$ and $g'(z_0) \neq 0$. **Warning:** this formula is **not valid** if $g(z_0) \neq 0$ or if $g'(z_0) = 0$.

22. Be able to use complex techniques to compute real integrals similar to those that have appeared in the homework problems.
23. Be able to compute Taylor series and Laurent series expansions (in **simple cases)** of a function f centered at a point $z_0 \in \mathbb{C}$. **Hint:** If $z_0 \neq 0$, write $z = z_0 + h$ and then do the expansion in h about $h = 0$. At the end replace h by $z - z_0$.