Inverse Functions

Definition 15.1 (Inverse Functions). If \( f : D \to \mathbb{C} \) is a complex function then we define \( f^{-1} : f(D) \to D \) by

\[
f^{-1}(z) = \{ w \in D : f(w) = z \}
\]

that is \( f^{-1}(z) \) is the set of all solutions for the equation \( f(w) = z \). In general this is not a function but a “multi-valued” function.

In general this is a multi-valued function and we will have to choose a branch when we need an honest function.

Definition 15.2 (Branches). Given a multi-valued function \( F : D \to \mathbb{C} \), we say a \( f : D_0 \subset D \to \mathbb{C} \) is a branch of \( F \) if \( f(z) \in F(z) \) for all \( z \in D_0 \) and \( f \) is continuous on \( D_0 \). Here \( D_0 \) is taken to be an open subset of \( D \).

Remark 15.3. Showed some simple pictures in class of functions on finite sets and then went over \( f(x) = x^2 \) for \( x \in \mathbb{R} \) so that \( f^{-1}(y) = \pm \sqrt{y} \) with \( \sqrt{y} \) being a chosen branch of \( f^{-1}(y) \).

Example 15.4. The “function” \( \text{arg} : D := \mathbb{C} \setminus \{0\} \to \mathbb{R} \subset \mathbb{C} \) is a multi-valued function and \( \text{Arg} : D_0 \to \mathbb{C} \) is a branch of \( \text{arg} \) where \( D_0 := \mathbb{C} \setminus (-\infty,0] \). Recall that \( \text{Arg}(z) = \theta \in (-\pi,\pi) \) such that \( z = |z|e^{i\theta} \). One may check that \( \text{Arg} \) is continuous on \( D_0 \) but not on \( D \) which is the reason for making the “branch cut.” To see that \( \text{Arg}(z) \) is continuous, let

\[
A(z) := \tan^{-1}(\text{Im } z / \text{Re } z) \text{ for } \text{Re } z > 0
\]

in which case observe that

\[
\text{Arg}(z) = A(z) := \tan^{-1}(\text{Im } z / \text{Re } z) \text{ for } \text{Re } z > 0
\]

which is continuous. If \( \text{Im } z > 0 \), then \( \text{Re } (-iz) > 0 \) and

\[
\text{Arg}(z) = \text{Arg}(-iz) + \pi/2 = A(-iz) + \pi/2 \text{ for } \text{Im } z > 0.
\]

Similarly, if \( \text{Im } z < 0 \), then \( \text{Re } (iz) > 0 \) and

\[
\text{Arg}(z) = \text{Arg}(iz) - \pi/2 = A(iz) - \pi/2 \text{ for } \text{Im } z > 0.
\]

In summary we have,

\[
\text{Arg}(z) = A(z) := \tan^{-1}(\text{Im } z / \text{Re } z) \text{ for } \text{Re } z > 0
\]

\[
\text{Arg}(z) = \frac{\pi}{2} - \tan^{-1}(\text{Re } z / \text{Im } z) \text{ for } \text{Im } z > 0, \text{ and}
\]

\[
\text{Arg}(z) = -\frac{\pi}{2} - \tan^{-1}(\text{Re } z / \text{Im } z) \text{ for } \text{Im } z < 0.
\]

Definition 15.5 (log). Given \( z \in \mathbb{C}^x \) we let \( \text{log}(z) = \exp^{-1}(z) = \{ w \in \mathbb{C} : e^w = z \} \) so that \( \text{log} \) is the inverse “function” to \( z \to e^z \).

Lemma 15.6 (log). The function \( \text{log} : \mathbb{C} \setminus \{0\} \to \mathbb{C} \) is given by

\[
\text{log} (z) = \ln |z| + i \text{Arg}(z) = \ln |z| + i \text{Arg}(z) + 2\pi i \mathbb{Z}.
\]

Proof. Writing \( w = a + ib \) and \( z = |z|e^{i\theta} \) we will have \( z = e^w \) iff \( e^a e^{ib} = |z|e^{i\theta} \). which happens iff \( |z| = e^a \) and \( b = \theta + 2\pi n \mathbb{Z} \), i.e. \( a = \ln |z| \) and \( b = \text{Arg}(z) \). Therefore \( w = \ln |z| + i \text{Arg}(z) = \text{log}(z) \).

Notation 15.7 (Log). For any \( z \neq 0 \) in \( \mathbb{C} \) we let

\[
\text{Log}(z) = \ln |z| + i \text{Arg}(z)
\]

so that \( \text{Log}(z) \in \log(z) \) for all \( z \in \mathbb{C}^x \). We refer to \( \text{Log} \) as the principle branch of \( \text{log} \).

\[
\text{Log}(r e^{i\theta}) = \ln r + i \theta \text{ provided } r > 0 \text{ and } -\pi < \theta \leq \pi.
\]

Example 15.8. If \( f(w) = w^n \) then \( f^{-1}(z) = z^{1/n} = |z|^{1/n} e^{i \frac{1}{n} \text{Arg}(z)} \). A branch of \( z^{1/n} \) can be found by taking

\[
z^{1/n} := |z|^{1/n} e^{i \frac{1}{n} \text{Arg}(z)}.
\]

Theorem 15.9 (Chain Rule Converse). Suppose \( g : U \subset \mathbb{C} \to \mathbb{C} \) is an analytic function and \( f : D \subset \mathbb{C} \to U \) is a branch of \( g^{-1} \), so that \( g(f(z)) = z \). Then \( f'(z) \) exists at all points \( z \in D \) so that \( g'(f(z)) \neq 0 \) and in this case

\[
f'(z) = \frac{1}{g'(f(z))}.
\]

[Note: assuming \( f'(z) \) exists we have by the chain rule that

\[
1 = \frac{d}{dz} z = \frac{d}{dz} g(f(z)) = g'(f(z)) f'(z)
\]

which helps to explain Eq. (15.1).]

Proof. Next class.
Inverse Functions II (2/12/2018)

Theorem 16.1 (Chain Rule Converse). Suppose \( g : U \subset_o \mathbb{C} \to \mathbb{C} \) is an analytic function and \( f : D \subset_o \mathbb{C} \to U \) is a branch of \( g^{-1} \), so that \( g(f(z)) = z \). Then \( f'(z) \) exists at all points \( z \in D \) so that \( g'(f(z)) \neq 0 \) and in this case
\[
f'(z) = \frac{1}{g'(f(z))}.
\]

[Note: assuming \( f'(z) \) exists we have by the chain rule that
\[
1 = \frac{d}{dz} z = \frac{d}{dz} g(f(z)) = g'(f(z)) f'(z)
\]
which helps to explain Eq. (15.1).]

Proof. Suppose that \( z \in D \) and \( g'(f(z)) \neq 0 \) and let \( \Delta f = f(z + h) - f(z) \). Using this notation we find,
\[
h = z + h - z = g(f(z + h)) - g(f(z)) = g'(f(z)) (f(z + h) - f(z)) = [g'(f(z)) + \varepsilon (\Delta f)] \Delta f.
\]

Therefore,
\[
\lim_{h \to 0} \frac{\Delta f}{h} = \lim_{h \to 0} \frac{1}{g'(f(z)) + \varepsilon (\Delta f)} = \frac{1}{g'(f(z))}
\]
wherein we have used the continuity of \( f \) at \( z \) in order to see \( \lim_{h \to 0} \Delta f = 0 \) and hence \( \lim_{h \to 0} \varepsilon (\Delta f) = 0 \).

16.1 More General converse to the Chain Rule (Not covered in class.)

Theorem 16.2 (Converse to the Chain Rule). Suppose \( f : D \subset_o \mathbb{C} \to U \subset_o \mathbb{C} \) and \( g : U \subset_o \mathbb{C} \to \mathbb{C} \) are functions such that \( f \) is continuous, \( g \) is analytic and \( \psi := g \circ f \) is analytic, then \( f \) is analytic on the set \( D \setminus \{ z : g'(f(z)) = 0 \} \). Moreover \( f'(z) = \psi'(z)/g'(f(z)) \) when \( z \in D \) and \( g'(f(z)) \neq 0 \).

Proof. Suppose that \( z \in D \) and \( g'(f(z)) \neq 0 \). Letting \( \Delta f = f(z + h) - f(z) \) and \( \Delta \psi := \psi(z + h) - \psi(z) \), we have
\[
\Delta \psi = g(f(z + \Delta f) - g(f(z)) = [g'(f(z)) + \varepsilon (\Delta f)] \Delta f.
\]
Dividing this equation by \( h \), solving for \( \Delta f/h \), and then letting \( h \to 0 \) shows
\[
\frac{\Delta f}{h} = \frac{1}{g'(f(z)) + \varepsilon (\Delta f)} \frac{\Delta \psi}{h} \to \frac{1}{g'(f(z))} \psi'(z).
\]
Here we have used the continuity of \( f \) at \( z \) so that \( \Delta f \to 0 \) as \( h \to 0 \) and therefore \( \lim_{h \to 0} (g'(f(z)) + \varepsilon (\Delta f)) = g'(f(z)) \neq 0 \).

Corollary 16.3 (Inverse function differentiation). Suppose that \( g : \mathbb{C} \to \mathbb{C} \) is an analytic function (\( g \) need not be defined on all of \( \mathbb{C} \)) and \( f : D \to \mathbb{C} \) is a continuous branch of \( g^{-1} \) so that \( g(f(z)) = z \) for all \( z \in D \). Then \( f'(z) \) exists at all \( z \in D \) where \( g'(f(z)) \neq 0 \) and at such \( z \),
\[
f'(z) = \frac{1}{g'(f(z))}.
\]
In other words one differentiates the equation \( g(f(z)) = z \) and solves the result for \( f'(z) \) where possible.

Proof. Apply Theorem 16.2 with \( h(z) = z \).

16.2 Differentiating Logarithms, \( n^{th} \) - Roots, inverse trig. functions

Example 16.4. We know that \( \log(z) \) is a branch of \( \log(z) \) and since \( e^{\log(z)} = z \) it follows from Theorem 16.1 or Corollary 16.3 that
\[
1 = \frac{d}{dz} z = \frac{d}{dz} e^{\log(z)} = e^{\log(z)} \frac{d}{dz} \log(z) = \frac{d}{dz} \log(z),
\]
i.e.
More generally: if \( f : D \to \mathbb{C} \) is any continuous branch of \( \log(z) \), then the exact same argument shows

\[
\ell'(z) = \frac{1}{z}.
\]

**Example 16.5 (CR proof that \( \log(z) \) is analytic (done in an earlier class)).** Let \( D := \mathbb{C} \setminus (-\infty,0] \) and take \( f(z) = \log(z) \) for \( z \in D \). Then \( f \) is an analytic function and

\[
\frac{d}{dz} \log(z) = \frac{1}{z} \quad \text{for all } z \in D.
\]

Let us check this for \( z = x + iy \) with \( x > 0 \) first. In this case

\[
f(z) = \log(z) = \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1}(y/x).
\]

Recall that \( \frac{d}{dt} \tan^{-1}(t) = \frac{1}{t^2 + 1} \), so we learn

\[
f_x = \frac{1}{2} \frac{2x}{x^2 + y^2} + i \frac{-y}{1 + (y/x)^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} = \frac{1}{|z|^2} \frac{x}{z} = \frac{1}{z}
\]

\[
f_y = \frac{1}{2} \frac{2y}{x^2 + y^2} + i \frac{1}{1 + (y/x)^2} = \frac{y}{x^2 + y^2} + i \frac{x}{x^2 + y^2} = \frac{i}{z} = if_x
\]

from which it follows that \( f \) is complex differentiable and \( f'(z) = \frac{1}{z} \).

Note that for \( \text{Im } z > 0 \), we have \( \log(z) = f \left( \frac{1}{z} \right) + i\pi/2 \) which shows \( \log(z) \) is complex differentiable for \( \text{Im } z > 0 \) and

\[
\frac{d}{dz} \log(z) = \frac{1}{i} f' \left( \frac{1}{i} z \right) = \frac{1}{i} \frac{1}{i} \frac{1}{z} = \frac{1}{z}.
\]

Similarly, if \( \text{Im } z < 0 \), we have \( \log(z) = f(iz) - i\pi/2 \) which shows \( \log(z) \) is complex differentiable for \( \text{Im } z < 0 \) and

\[
\frac{d}{dz} \log(z) = if' \left( iz \right) = \frac{1}{iz} = \frac{1}{z}.
\]

Combining these remarks shows that \( \log(z) \) is complex differentiable on \( \mathbb{C} \setminus (-\infty,0] \). (We will give another simpler proof using the converse to the chain rule, see Theorem 16.2 below.)

**Example 16.6.** Suppose that \( \rho : D \to \mathbb{C} \) is branch of

\[
z^{1/n} = \{ w : w^n = z \} = f_n^{-1}(z)
\]

where \( f_n(z) = z^n \). Then differentiating \( \rho^n(z) = z \) gives

\[
\rho'(z) = \frac{1}{n} \rho^{n-1}(z) \rho'(z) = \frac{1}{n} z^{1/n} \rho(z) = \frac{1}{n} \rho(z).
\]

so that

\[
\rho'(z) = \frac{1}{n} \rho^{n-1}(z) = \frac{1}{n} \rho^n(z) = \frac{1}{n} \rho(z).
\]

Formally this states,

\[
\frac{d}{dz} z^{1/n} = \frac{1}{n} z^{1/n} - \frac{1}{n} z^{-1/n}.
\]

**Remark 16.7.** For example we might take

\[
\rho(z) = |z|^{1/2} e^{i \frac{1}{n} \arg(z)} \quad \text{for } z \in D_0 := \mathbb{C} \setminus (-\infty,0].
\]

This can be written as \( \rho(z) = e^{i \frac{1}{n} \log(z)} \) in \( z^{1/n} \) and so we may compute

\[
\rho'(z) = e^{i \frac{1}{n} \log(z)} \cdot \frac{d}{dz} \frac{1}{n} \log(z) = \frac{1}{n} \rho(z)
\]

as has to hold in general.

**Example 16.8.** Suppose that \( f(z) \) is a branch of \( \tan^{-1}(z) \), i.e. \( \tan(f(z)) = z \).

Differentiating this using

\[
\frac{d}{dw} \tan(w) = \frac{d}{dw} \sin(w) = \frac{1}{\cos^2(w)}
\]

equation shows,

\[
1 = \frac{1}{\cos^2(f(z))} f'(z) \iff f'(z) = \cos^2(f(z)).
\]

Now

\[
z^2 = \tan^2(f(z)) = \frac{\sin^2(f(z))}{\cos^2(f(z))} = \frac{1 - \cos^2(f(z))}{\cos^2(f(z))}
\]

from which it follows that

\[
\cos^2(f(z)) = \frac{1}{1 + z^2}
\]

and therefore,

\[
f'(z) = \frac{1}{1 + z^2}.
\]

That is

\[
\frac{d}{dz} \tan^{-1}(z) = \frac{1}{1 + z^2}.
\]

This equation is valid for any branch of \( \tan^{-1}(z) \)!
Example 16.9. Suppose that \( f(z) \) is a branch of \( \sin^{-1}(z) \), i.e. \( \sin(f(z)) = z \). Differentiating this equation shows,

\[
1 = \cos(f(z)) f'(z) \quad \Rightarrow \quad f'(z) = \frac{1}{\cos(f(z))}
\]

provided \( \cos(f(z)) \neq 0 \). Let us further observe that \( \cos^2(f(z)) = 1 - \sin^2(f(z)) = 1 - z^2 \) and hence \( \cos(f(z)) \in (1 - z^2)^{1/2} \). Thus modulo delicate branch issues we expect,

\[
\frac{d}{dz} \sin^{-1}(z) = (1 - z^2)^{-1/2}.
\]

16.3 Power Functions

Definition 16.10 \((z^c)\). For general \( c \in \mathbb{C} \) we define for \( z \neq 0 \),

\[
z^c := e^{c \log z} \quad \text{for all } z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}.
\]

In detail if \( z = re^{i\theta} \) then \( \log z = \ln r + i(\theta + 2\pi k) \) and

\[
z^c = \left\{ e^{c(\ln r + i(\theta + 2\pi k))} : k \in \mathbb{Z} \right\}.
\]

Example 16.11. Let \( \ell \) be a branch of \( \log(z) \), i.e. a continuous choice \( \ell : D \to \mathbb{C} \) such that \( \ell(z) \in \log(z) \) for all \( z \in D \) then we define \( z_\ell^c := e^{c\ell(z)} \) and with this definition,

\[
\frac{d}{dz} z_\ell^c = \frac{d}{dz} e^{c\ell(z)} = e^{c\ell(z)} c\ell'(z)
\]

\[
= ce^{c\ell(z)} \frac{1}{z} = ce^{c\ell(z)} e^{-\ell(z)} = ce^{(c-1)\ell(z)} = cz_\ell^{c-1}.
\]