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Inverse Functions

Definition 15.1 (Inverse Functions). *If $f : D \to \mathbb{C}$ is a complex function then we define $f^{-1} : f(D) \rightsquigarrow D$ by*

$$f^{-1}(z) = \{w \in D : f(w) = z\}$$

that is $f^{-1}(z)$ is the set of all solutions for the equation $f(w) = z$. In general this not a function but a "multi-valued" function.

In general this is a multi-valued function and we will have to choose a branch when we need an honest function.

Definition 15.2 (Branches). *Given a multi-valued function $F : D \to \mathbb{C}$, we say a $f : D_0 \subset D \to \mathbb{C}$ is a* ***branch*** *of F if $f(z) \in F(z)$ for all $z \in D_0$ and f is continuous on D_0. Here D_0 is taken to be an open subset of D.*

Remark 15.3. Showed some simple pictures in class of functions on finite sets and then went over $f(x) = x^2$ for $x \in \mathbb{R}$ so that $f^{-1}(y) = \pm\sqrt{y}$ with $\sqrt{y}$ being a chosen branch of $f^{-1}(y)$.

Example 15.4. The "function" $\arg : D := \mathbb{C} \setminus \{0\} \rightsquigarrow \mathbb{R} \subset \mathbb{C}$ is a multi-valued function and $\operatorname{Arg} : D_0 \to \mathbb{C}$ is a branch of arg where $D_0 := \mathbb{C} \setminus (-\infty, 0]$. Recall that $\operatorname{Arg}(z) = \theta \in (-\pi, \pi)$ such that $z = |z| e^{i\theta}$. One may check that Arg is continuous on D_0 but not on D which is the reason for making the "**branch cut**." To see that $\operatorname{Arg}(z)$ is continuous, let

$$A(z) := \tan^{-1}(\operatorname{Im} z / \operatorname{Re} z) \text{ for } \operatorname{Re} z > 0$$

in which case observe that

$$\operatorname{Arg}(z) = A(z) := \tan^{-1}(\operatorname{Im} z / \operatorname{Re} z) \text{ for } \operatorname{Re} z > 0$$

which is continuous. If $\operatorname{Im} z > 0$, then $\operatorname{Re}(-iz) > 0$ and

$$\operatorname{Arg}(z) = \operatorname{Arg}(-iz) + \pi/2 = A(-iz) + \pi/2 \text{ for } \operatorname{Im} z > 0.$$

Similarly, if $\operatorname{Im} z < 0$, then $\operatorname{Re}(iz) > 0$ and

$$\operatorname{Arg}(z) = \operatorname{Arg}(iz) - \pi/2 = A(iz) - \pi/2 \text{ for } \operatorname{Im} z > 0.$$

In summary we have,

$$\begin{aligned}
\operatorname{Arg}(z) &= A(z) := \tan^{-1}(\operatorname{Im} z / \operatorname{Re} z) \text{ for } \operatorname{Re} z > 0 \\
\operatorname{Arg}(z) &= \frac{\pi}{2} - \tan^{-1}(\operatorname{Re} z / \operatorname{Im} z) \text{ for } \operatorname{Im} z > 0, \text{ and} \\
\operatorname{Arg}(z) &= -\frac{\pi}{2} - \tan^{-1}(\operatorname{Re} z / \operatorname{Im} z) \text{ for } \operatorname{Im} z < 0.
\end{aligned}$$

Definition 15.5 (log). *Given $z \in \mathbb{C}^\times$ we let $\log(z) = \exp^{-1}(z) = \{w \in \mathbb{C} : e^w = z\}$ so that* log *is the inverse "function" to $z \to e^z$.*

Lemma 15.6 (log). *The function $\log : \mathbb{C} \setminus \{0\} \rightsquigarrow \mathbb{C}$ is given by*

$$\log(z) = \ln|z| + i \arg(z) = \ln|z| + i \operatorname{Arg}(z) + 2\pi i \mathbb{Z}.$$

Proof. Writing $w = a + ib$ and $z = |z| e^{i\theta}$ we will have $z = e^w$ iff $e^a e^{ib} = |z| e^{i\theta}$ which happens iff $|z| = e^a$ and $b = \theta + 2\pi i \mathbb{Z}$, i.e. $a = \ln|z|$ and $b = \arg(z)$. Therefore $w = \ln|z| + i \arg(z) = \log(z)$. ■

Notation 15.7 (Log) *For any $z \neq 0$ in $\mathbb{C}$ we let*

$$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z)$$

so that $\operatorname{Log}(z) \in \log(z)$ for all $z \in \mathbb{C}^\times$. We refer to Log *as the principle* ***branch*** *of* log.

$$\operatorname{Log}(re^{i\theta}) = \ln r + i\theta \text{ provided } r > 0 \text{ and } -\pi < \theta \leq \pi.$$

Example 15.8. If $f(w) = w^n$ then $f^{-1}(z) = z^{1/n} = |z|^{\frac{1}{n}} e^{i\frac{1}{n}\arg(z)}$. A brach of $z^{1/n}$ can be found by taking

$$z_{PV}^{1/n} := |z|^{\frac{1}{n}} e^{i\frac{1}{n}\operatorname{Arg}(z)}.$$

Theorem 15.9 (Chain Rule Converse). *Suppose $g : U \subset_o \mathbb{C} \to \mathbb{C}$ is an analytic function and $f : D \subset_o \mathbb{C} \to U$ is a branch of g^{-1}, so that $g(f(z)) = z$. Then $f'(z)$ exists at all points $z \in D$ so that $g'(f(z)) \neq 0$ and in this case*

$$f'(z) = \frac{1}{g'(f(z))}. \tag{15.1}$$

[Note: assuming $f'(z)$ exists we have by the chain rule that

$$1 = \frac{d}{dz} z = \frac{d}{dz} g(f(z)) = g'(f(z)) f'(z)$$

which helps to explain Eq. (15.1).]

Proof. Next class. ■

Inverse Functions II (2/12/2018)

Theorem 16.1 (Chain Rule Converse). *Suppose $g : U \subset_o \mathbb{C} \to \mathbb{C}$ is an analytic function and $f : D \subset_o \mathbb{C} \to U$ is a branch of g^{-1}, so that $g(f(z)) = z$. Then $f'(z)$ exists at all points $z \in D$ so that $g'(f(z)) \neq 0$ and in this case*

$$f'(z) = \frac{1}{g'(f(z))}. \tag{16.1}$$

[Note: assuming $f'(z)$ exists we have by the chain rule that

$$1 = \frac{d}{dz}z = \frac{d}{dz}g(f(z)) = g'(f(z))\, f'(z)$$

which helps to explain Eq. (15.1).]

Proof. Suppose that $z \in D$ and $g'(f(z)) \neq 0$ and let $\Delta f = f(z+h) - f(z)$. Using this notation we find,

$$\begin{aligned} h = z + h - z &= g(f(z+h)) - g(f(z)) \\ &= g(f(z) + \Delta f) - g(f(z)) \\ &= [g'(f(z)) + \varepsilon(\Delta f)]\, \Delta f. \end{aligned}$$

Therefore,

$$\lim_{h\to 0} \frac{\Delta f}{h} = \lim_{h\to 0} \frac{1}{g'(f(z)) + \varepsilon(\Delta f)} = \frac{1}{g'(f(z))}$$

wherein we have use the continuity of f at z in order to see $\lim_{h\to 0} \Delta f = 0$ and hence $\lim_{h\to 0} \varepsilon(\Delta f) = 0$. ■

16.1 More General converse to the Chain Rule (Not covered in class.)

Theorem 16.2 (Converse to the Chain Rule). *Suppose $f : D \subset_o \mathbb{C} \to U \subset_o \mathbb{C}$ and $g : U \subset_o \mathbb{C} \to \mathbb{C}$ are functions such that f is continuous, g is analytic and $\psi := g \circ f$ is analytic, then f is analytic on the set $D \setminus \{z : g'(f(z)) = 0\}$. Moreover $f'(z) = \psi'(z)/g'(f(z))$ when $z \in D$ and $g'(f(z)) \neq 0$.*

Proof. Suppose that $z \in D$ and $g'(f(z)) \neq 0$. Letting $\Delta f = f(z+h) - f(z)$ and $\Delta\psi := \psi(z+h) - \psi(z)$, we have

$$\Delta\psi = g(f(z) + \Delta f) - g(f(z)) = [g'(f(z)) + \varepsilon(\Delta f)]\, \Delta f.$$

Dividing this equation by h, solving for $\Delta f/h$, and then letting $h \to 0$ shows

$$\frac{\Delta f}{h} = \frac{1}{g'(f(z)) + \varepsilon(\Delta f)} \frac{\Delta\psi}{h} \to \frac{1}{g'(f(z))}\psi'(z).$$

Here we have used the continuity of f at z so that $\Delta f \to 0$ as $h \to 0$ and therefore $\lim_{h\to 0}(g'(f(z)) + \varepsilon(\Delta f)) = g'(f(z)) \neq 0$.

■

Corollary 16.3 (Inverse function differentiation). *Suppose that $g : \mathbb{C} \to \mathbb{C}$ is an analytic function (g need not be defined on all of $\mathbb{C}$) and $f : D \to \mathbb{C}$ is a continuous branch of g^{-1} so that $g(f(z)) = z$ for all $z \in D$. Then $f'(z)$ exists at all $z \in D$ where $g'(f(z)) \neq 0$ and at such z,*

$$f'(z) = \frac{1}{g'(f(z))}.$$

In other words one differentiates the equation $g(f(z)) = z$ and solves the result for $f'(z)$ where possible.

Proof. Apply Theorem 16.2 with $h(z) = z$. ■

16.2 Differentiating Logarithms, n^{th} - Roots, inverse trig. functions

Example 16.4. We know that $\mathrm{Log}(z)$ is a branch of $\log(z)$ and since $e^{\mathrm{Log}(z)} = z$ it follows from Theorem 16.1 or Corollary 16.3 that

$$1 = \frac{d}{dz}z = \frac{d}{dz}e^{\mathrm{Log}(z)} = e^{\mathrm{Log}(z)}\frac{d}{dz}\mathrm{Log}(z) = z\frac{d}{dz}\mathrm{Log}(z),$$

i.e.

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z}.$$

More generally: if $\ell : D \to \mathbb{C}$ is any continuous branch of $\log(z)$, then the exact same argument shows

$$\ell'(z) = \frac{1}{z}.$$

Example 16.5 (CR proof that $\mathrm{Log}(z)$ *is analytic (done in an earlier class)).* Let $D := \mathbb{C} \setminus (-\infty, 0]$ and take $f(z) = \mathrm{Log}(z)$ for $z \in D$. Then f is an analytic function and

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{z} \text{ for all } z \in D.$$

Let us check this for $z = x + iy$ with $x > 0$ first. In this case

$$f(z) = \mathrm{Log}(z) = \frac{1}{2}\ln\left(x^2 + y^2\right) + i\tan^{-1}(y/x).$$

Recall that $\frac{d}{dt}\tan^{-1}(t) = \frac{1}{t^2+1}$ so we learn

$$f_x = \frac{1}{2}\frac{2x}{x^2+y^2} + i\frac{-\frac{y}{x^2}}{1+(y/x)^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{1}{|z|^2}\bar{z} = \frac{1}{z}$$
$$f_y = \frac{1}{2}\frac{2y}{x^2+y^2} + i\frac{\frac{1}{x}}{1+(y/x)^2} = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2} = i\frac{1}{z} = if_x$$

from which it follows that f is complex differentiable and $f'(z) = \frac{1}{z}$.

Note that for $\mathrm{Im}\, z > 0$, we have $\mathrm{Log}(z) = f\left(\frac{1}{i}z\right) + i\pi/2$ which shows $\mathrm{Log}(z)$ is complex differentiable for $\mathrm{Im}\, z > 0$ and

$$\frac{d}{dz}\mathrm{Log}(z) = \frac{1}{i}f'\left(\frac{1}{i}z\right) = \frac{1}{i}\frac{1}{\frac{1}{i}z} = \frac{1}{z}.$$

Similarly, if $\mathrm{Im}\, z < 0$, we have $\mathrm{Log}(z) = f(iz) - i\pi/2$ which shows $\mathrm{Log}(z)$ is complex differentiable for $\mathrm{Im}\, z < 0$ and

$$\frac{d}{dz}\mathrm{Log}(z) = if'(iz) = i\frac{1}{iz} = \frac{1}{z}.$$

Combining these remarks shows that $\mathrm{Log}(z)$ is complex differentiable on $\mathbb{C} \setminus (-\infty, 0]$. (We will give another simpler proof using the converse to the chain rule, see Theorem 16.2 below.)

Example 16.6. Suppose that $\rho : D \to \mathbb{C}$ is branch of

$$z^{1/n} = \{w : w^n = z\} = f_n^{-1}(z)$$

where $f_n(z) = z^n$. Then differentiating $\rho^n(z) = z$ gives

$$n\rho^{n-1}(z)\rho'(z) = 1$$

so that

$$\rho'(z) = \frac{1}{n}\frac{1}{\rho^{n-1}(z)} = \frac{1}{n}\frac{\rho(z)}{\rho^n(z)} = \frac{1}{n}\frac{\rho(z)}{z}.$$

Formally this states,

$$\text{“}\frac{d}{dz}z^{1/n} = \frac{1}{n}\frac{z^{1/n}}{z} = \frac{1}{n}z^{1/n-1}.\text{”}$$

Remark 16.7. For example we might take

$$\rho(z) = |z|^{\frac{1}{n}} e^{i\frac{1}{n}\mathrm{Arg}(z)} \text{ for } z \in D_0 := \mathbb{C} \setminus (-\infty, 0].$$

This can be written as $\rho(z) = e^{\frac{1}{n}\mathrm{Log}(z)} \in z^{1/n}$ and so we may compute

$$\rho'(z) = e^{\frac{1}{n}\mathrm{Log}(z)} \cdot \frac{d}{dz}\frac{1}{n}\mathrm{Log}(z) = \frac{1}{n}\frac{\rho(z)}{z}$$

as has to hold in general.

Example 16.8. Suppose that $f(z)$ is a branch of $\tan^{-1}(z)$, i.e. $\tan(f(z)) = z$. Differentiating this using

$$\frac{d}{dw}\tan(w) = \frac{d}{dw}\frac{\sin(w)}{\cos(w)} = \frac{1}{\cos^2(w)}$$

equation shows,

$$1 = \frac{1}{\cos^2(f(z))}f'(z) \implies f'(z) = \cos^2(f(z)).$$

Now

$$z^2 = \tan^2(f(z)) = \frac{\sin^2(f(z))}{\cos^2(f(z))} = \frac{1 - \cos^2(f(z))}{\cos^2(f(z))}$$

from which it follows that

$$\cos^2(f(z)) = \frac{1}{1+z^2}$$

and therefore,

$$f'(z) = \frac{1}{1+z^2}.$$

That is

$$\text{“}\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}.\text{”}$$

This equation is valid for any branch of $\tan^{-1}(z)$!

Example 16.9. Suppose that $f(z)$ is a branch of $\sin^{-1}(z)$, i.e. $\sin(f(z)) = z$. Differentiating this equation shows,

$$1 = \cos(f(z))\, f'(z) \implies f'(z) = \frac{1}{\cos(f(z))}$$

provided $\cos(f(z)) \neq 0$. Let us further observe that $\cos^2(f(z)) = 1 - \sin^2(f(z)) = 1 - z^2$ and hence $\cos(f(z)) \in \left(1 - z^2\right)^{1/2}$. Thus modulo delicate branch issues we expect,

$$\text{“}\frac{d}{dz}\sin^{-1}(z) = \left(1 - z^2\right)^{-1/2}.\text{”}$$

16.3 Power Functions

Definition 16.10 (z^c). *For general $c \in \mathbb{C}$ we define for $z \neq 0$,*

$$z^c := e^{c \log z} \text{ for all } z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}.$$

In detail if $z = re^{i\theta}$ then $\log z = \ln r + i(\theta + 2\pi\mathbb{Z})$ and

$$z^c = \left\{ e^{c(\ln r + i(\theta + 2\pi k))} : k \in \mathbb{Z} \right\}.$$

Example 16.11. Let ℓ be a **branch of** $\log(z)$, i.e. a continuous choice $\ell : D \to \mathbb{C}$ such that $\ell(z) \in \log(z)$ for all $z \in D$ then we define $z_\ell^c := e^{c\ell(z)}$ and with this definition,

$$\begin{aligned} \frac{d}{dz} z_\ell^c &= \frac{d}{dz} e^{c\ell(z)} = e^{c\ell(z)} c\ell'(z) \\ &= c e^{c\ell(z)} \frac{1}{z} = c e^{c\ell(z)} e^{-\ell(z)} = c e^{(c-1)\ell(z)} = c z_\ell^{c-1}. \end{aligned}$$