Study Guide for Math 120A Midterm 1 (Wednesday, January 31, 2018)

1. \( \mathbb{C} := \{ z = x + iy : x, y \in \mathbb{R} \} \) with \( i^2 = -1 \) and \( \bar{z} = x - iy \). The complex numbers behave much like the real numbers. In particular the quadratic formula holds.

2. \( |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}, |zw| = |z||w|, |z + w| \leq |z| + |w|, \) \( \text{Re} z = \frac{z + \bar{z}}{2}, \)
   \( \text{Im} z = \frac{z - \bar{z}}{2i}, |\text{Re} z| \leq |z| \) and \( |\text{Im} z| \leq |z| \).

3. We also have \( z\bar{w} = \bar{z}w \) and \( z + \bar{w} = \bar{z} + w \) and \( z^{-1} = \frac{\bar{z}}{|z|^2} \).

4. \( \{ z : |z - z_0| = \rho \} \) is a circle of radius \( \rho \) centered at \( z_0 \).

5. \( e^z = e^x (\cos y + i \sin y) \), every \( z = |z| e^{i\theta} \).

6. Know that \( e^z e^w = e^{z+w} \) for all \( z, w \in \mathbb{C} \).

7. \( |e^z| = e^x \leq e^{|z|} \).

8. \( \text{arg} (z) = \{ \theta \in \mathbb{R} : z = |z| e^{i\theta} \} \) and \( \text{Arg} (z) = \theta \) if \(-\pi < \theta \leq \pi \) and \( z = |z| e^{i\theta} \). Notice that \( z = |z| e^{i\text{arg}(z)} \).

9. \( z^{1/n} = \sqrt[n]{|z|} e^{\frac{i\text{Arg}(z)}{n}} = \sqrt[n]{|z|} \left\{ e^{\frac{i\text{Arg}(z)+2\pi k}{n}} : k = 0, 1, 2, \ldots , n - 1 \right\} \).

10. Differentiation of complex functions, \( z(t) = a(t) + i b(t) \), of one real variable \( t \). Recall that \( \dot{z}(t) = \dot{a}(t) + i \dot{b}(t) \)
    and that the usual rules hold like
    \[ \frac{d}{dt} [z(t) + w(t)] = \dot{z}(t) + \dot{w}(t) \]
    and
    \[ \frac{d}{dt} [z(t) w(t)] = \dot{z}(t) w(t) + z(t) \dot{w}(t) \].

11. Integration:
    \[ \int_{a}^{b} z(t) \, dt := \int_{a}^{b} x(t) \, dt + i \int_{a}^{b} y(t) \, dt. \]
    All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.

12. Know how to estimate the size of integrals using,
    \[ \left| \int_{a}^{b} z(t) \, dt \right| \leq \int_{a}^{b} |z(t)| \, dt \leq M \cdot (b-a) \]
    where \( M \geq 0 \) is any constant such that \( |z(t)| \leq M \) for \( a \leq t \leq b \).

13. Know how to compute simple limits \( \lim_{z \to z_0} f(z) = L \) where \( z_0, L \in \mathbb{C} \cup \{ \infty \} \). (Usual limit rules hold from real variables.)