Study Guide for Math 120A Midterm 1 (Monday, October 19, 2015)

1. \( \mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\} \) with \( i^2 = -1 \) and \( \bar{z} = x - iy \). The complex numbers behave much like the real numbers. In particular the quadratic formula holds.

2. \( |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}, \quad |zw| = |z||w|, \quad |z + w| \leq |z| + |w|, \quad \text{Re} z = \frac{z + \bar{z}}{2}, \quad \text{Im} z = \frac{z - \bar{z}}{2i}, \quad |\text{Re} z| \leq |z| \) and \( |\text{Im} z| \leq |z| \).

3. We also have \( \mathbb{C} \) and \( \mathbb{C} + \mathbb{C} = \mathbb{C} \). As \( |z| \) and \( |z + w| = |\bar{z} + \bar{w}| \) and \( z^{-1} = \frac{\bar{z}}{z^2} \).

4. \( \{z : |z - z_0| = \rho \} \) is a circle of radius \( \rho \) centered at \( z_0 \).

5. \( e^z = e^x \cos y + i e^x \sin y \), every \( z = |z| \ e^{i\theta} \).

6. \( |e^z| = e^{|z|} \leq e^{|z|} \).

7. \( \arg (z) = \{\theta \in \mathbb{R} : z = |z| e^{i\theta}\} \) and \( \Arg (z) = \theta \) if \( -\pi < \theta \leq \pi \) and \( z = |z| e^{i\theta} \). Notice that \( z = |z| e^{i\Arg(z)} \).

8. \( z^{1/n} = \sqrt[n]{|z|} e^{i\frac{\Arg(z)}{n}} \) \( = \sqrt[n]{|z|} \left\{ e^{i\frac{\Arg(z) + 2k\pi}{n}} : k = 0, 1, 2, \ldots, n - 1 \right\} \).

9. Differentiation of complex functions, \( z(t) = a(t) + ib(t) \), of one real variable \( t \). Recall that

\[ \frac{d}{dt} [z(t) + w(t)] = \dot{z}(t) + \dot{w}(t) \]

and that the usual rules hold like

\[ \frac{d}{dt} [z(t) + w(t)] = \dot{z}(t) + \dot{w}(t) \]

and

\[ \frac{d}{dt} [z(t) w(t)] = \dot{z}(t) w(t) + z(t) \dot{w}(t) \).

10. Integration:

\[ \int_a^b z(t) \, dt := \int_a^b x(t) \, dt + i \int_a^b y(t) \, dt. \]

All of the usual integration rules hold, like the fundamental theorem of calculus, linearity and integration by parts.


12. Know how to compute simple limits \( \lim_{z \to z_0} f(z) = L \) where \( z_0, L \in \mathbb{C} \cup \{\infty\} \). (Usual limit rules hold from real variables.)

13. The definition of complex differentiability. Examples, \( p(z), e^z, e^{p(z)}, 1/z, 1/p(z) \) etc.

14. Key points of \( e^z \) are \( \frac{d}{dz} e^z = e^z \) and \( e^z e^w = e^{z+w} \).

15. All of the usual derivative formulas hold, in particular product, sum, and chain rules:

\[ \frac{d}{dz} f(g(z)) = f'(g(z))g'(z) \]

and

\[ \frac{d}{dt} f(z(t)) = f'(z(t)) \dot{z}(t) \]

where \( f, g \) are analytic and \( z(t) \in \mathbb{C} \) is continuously differentiable.

16. \( \Re z, \Im z, \bar{z} \), are nice functions from the real - variables point of view but are not complex differentiable.

17. The Cauchy Riemann (C.R.) equations hold,

\[ f_y = i f_x \text{ or equivalently if } f = u + iv \text{ then } u_y = -v_x \text{ and } u_x = v_y \]

if \( f \) is complex differentiable. Conversely, if the C.R. equations hold and the partial derivatives are continuous near some point \( z \) then \( f'(z) \) exists and \( f'(z) = f_x(z) = -i f_y(z) \).