1. Math 120B Homework #9 Added Problems

Let $D = D(0,1)$ be the unit disk centered at $0 \in \mathbb{C}$ and

$$C = \partial D = \{z = x + iy \in \mathbb{C} : |z| = 1\}$$

be the unit circle in $\mathbb{C}$ which is the boundary of D. Our **goal** is to solve the following Dirichlet problem.

A Dirichlet problem on D. Given a polynomial function of (x, y),

$$p(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{N} c_{m,n} x^m y^n \text{ with } c_{m,n} \in \mathbb{R},$$

we wish to find a harmonic function, $u : D \to \mathbb{R}$ such that $u = p$ on the boundary of D, i.e. $\Delta u(x,y) = 0$ with

$$u(x,y) = p(x,y) \text{ when } |x + iy| = 1, \text{ i.e. for } z = x + iy \in C.$$

Here are three steps to follow in order solve this problem.

(1) By expressing x and y as

$$x = \frac{1}{2}(z + \bar{z}) \text{ and } y = \frac{1}{2i}(z - \bar{z})$$

we can find $C_{m,n} \in \mathbb{C}$ so that

$$p(x,y) = P(z, \bar{z}) = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{m,n} z^m \bar{z}^n. \tag{1.1}$$

(2) Since $z \cdot \bar{z} = 1$ for $z \in C := \partial D$, it follows that

$$z^m \bar{z}^n = \begin{cases} z^{m-n} & \text{if } m \geq n \\ \bar{z}^{n-m} & \text{if } n \geq m \end{cases} \text{ for } z \in C.$$

Using this result in Eq. (1.1), we can find $a_m, b_n \in \mathbb{C}$ so that

$$p(x,y) = \sum_{k=0}^{M} a_m z^k + \sum_{n=0}^{N} b_n \bar{z}^n \text{ for } z = x + iy \in C.$$

(3) From Exercise 1 below and the fact that real parts of analytic functions are harmonic, we conclude that

$$u(x,y) := \operatorname{Re}\left[\sum_{k=0}^{M} a_k z^k + \sum_{n=0}^{N} b_n \bar{z}^n\right] = \operatorname{Re}\sum_{k=0}^{M} a_k z^k + \operatorname{Re}\sum_{n=0}^{N} b_n \bar{z}^n,$$

is a harmonic function such that $u = p$ on $C = \partial D$.

Example 1. Let us find a harmonic function, $u(x,y)$, such that $u(x,y) = x^2$ when $|x+iy| = 1$. To do this we use, for $|z| = 1$, that

$$\begin{aligned} x^2 = \left(\frac{z+\bar{z}}{2}\right)^2 &= \frac{1}{4}\left[z^2 + 2z\ \bar{z} + \bar{z}^2\right] \\ &= \frac{1}{4}\left[z^2 + 2 + \bar{z}^2\right]. \end{aligned}$$

Hence the solution is

$$\begin{aligned} u(x,y) = \operatorname{Re}\frac{1}{4}\left[z^2 + \bar{z}^2 + 2\right] &= \frac{1}{4}\left[2\left(x^2 - y^2\right) + 2\right] \\ &= \frac{1}{2}\left[1 + x^2 - y^2\right]. \end{aligned}$$

Note that when $|z| = 1$ we have $1 - y^2 = x^2$ and so it is easy to see that $u(x,y) = x^2$ when $(x,y) \in C$. Moreover this function is easily seen to be harmonic.

Exercise 1. If $f : \mathbb{C} \to \mathbb{C}$ is analytic, show

$$u(x,y) = \operatorname{Re}\left[f(\bar{z})\right] = \operatorname{Re}\left[f(x - iy)\right]$$

is harmonic.

Exercise 2. Find a harmonic function, $u(x,y)$, such that $u(x,y) = p(x,y)$ when $|x+iy| = 1$ where;

(1) $p(x,y) = x^2 y$,
(2) $p(x,y) = x^3$.