Let $D = D(0, 1)$ be the unit disk centered at 0 $\in \mathbb{C}$ and $C = \partial D = \{z = x + iy \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$ which is the boundary of $D$. Our goal is to solve the following Dirichlet problem.

A Dirichlet problem on $D$. Given a polynomial function of $(x, y)$, $p(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} c_{m,n} x^m y^n$ with $c_{m,n} \in \mathbb{R}$, we wish to find a harmonic function, $u : D \rightarrow \mathbb{R}$ such that $u = p$ on the boundary of $D$, i.e. $\Delta u(x, y) = 0$ with

$$u(x, y) = p(x, y) \text{ when } |x + iy| = 1, \text{ i.e. for } z = x + iy \in C.$$

Here are three steps to follow in order solve this problem.

(1) By expressing $x$ and $y$ as

$$x = \frac{1}{2} (z + \bar{z}) \text{ and } y = \frac{1}{2i} (z - \bar{z})$$

we can find $C_{m,n} \in \mathbb{C}$ so that

$$p(x, y) = P(z, \bar{z}) = \sum_{m=0}^{M} \sum_{n=0}^{N} C_{m,n} z^m \bar{z}^n .$$

(2) Since $z \cdot \bar{z} = 1$ for $z \in C := \partial D$, it follows that

$$z^m \bar{z}^n = \begin{cases} z^{m-n} \text{ if } m \geq n & \text{if } n \geq m \text{ for } z \in C. \end{cases}$$

Using this result in Eq. (1.1), we can find $a_m, b_n \in \mathbb{C}$ so that

$$p(x, y) = \sum_{k=0}^{M} a_k z^k + \sum_{n=0}^{N} b_n \bar{z}^n \text{ for } z = x + iy \in C.$$

(3) From Exercise 1 below and the fact that real parts of analytic functions are harmonic, we conclude that

$$u(x, y) := \text{Re} \left[ \sum_{k=0}^{M} a_k z^k + \sum_{n=0}^{N} b_n \bar{z}^n \right] = \text{Re} \sum_{k=0}^{M} a_k z^k + \text{Re} \sum_{n=0}^{N} b_n \bar{z}^n,$$

is a harmonic function such that $u = p$ on $C = \partial D$. 

1. Math 120B Homework #9 Added Problems
Example 1. Let us find a harmonic function, $u(x, y)$, such that $u(x, y) = x^2$ when $|x + iy| = 1$. To do this we use, for $|z| = 1$, that

$$x^2 = \left(\frac{z + \bar{z}}{2}\right)^2 = \frac{1}{4} [z^2 + 2z \bar{z} + \bar{z}^2]$$

$$= \frac{1}{4} [z^2 + 2 + \bar{z}^2].$$

Hence the solution is

$$u(x, y) = \text{Re} \frac{1}{4} [z^2 + \bar{z}^2 + 2] = \frac{1}{4} [2 (x^2 - y^2) + 2]$$

$$= \frac{1}{2} [1 + x^2 - y^2].$$

Note that when $|z| = 1$ we have $1 - y^2 = x^2$ and so it is easy to see that $u(x, y) = x^2$ when $(x, y) \in C$. Moreover this function is easily seen to be harmonic.

Exercise 1. If $f : \mathbb{C} \to \mathbb{C}$ is analytic, show

$$u(x, y) = \text{Re} \left[ f(\bar{z}) \right] = \text{Re} \left[ f(x - iy) \right]$$

is harmonic.

Exercise 2. Find a harmonic function, $u(x, y)$, such that $u(x, y) = p(x, y)$ when $|x + iy| = 1$ where;

1. $p(x, y) = x^2y$,
2. $p(x, y) = x^3$. 