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4/9/2018

43.1 Classification of Singularities without Laurent

Theorem 43.1 (Classification of singularities without Laurent). *If $f : D'(z_0, \varepsilon) \to \mathbb{C}$ is an analytic function with an isolated singularity at z_0, then;*

1. *f has a removable singularity at z_0 iff f is bounded near z_0.*
2. *f has a pole at z_0 iff $\lim_{z\to z_0} f(z) = \infty$.*
3. *f has an essential singularity at z_0 iff $\lim_{z\to z_0} f(z)$ does not exists in $\mathbb{C}_\infty$ iff $f(D'(z_0,\delta))$ is dense in $\mathbb{C}$ for all $0 < \delta < \varepsilon$.*

[Item 3. is referred to as the Casorati–Weierstrass theorem.]

Proof. For notational simplicity we assume that $z_0 = 0$. If f has a removable singularity then $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and we see that if we define $f(0) = a_0$, then f is analytic at z_0 and hence bounded there. Conversely if f is bounded near 0, then for $n \in \mathbb{N}$,

$$|a_{-n}| = \left| \frac{1}{2\pi i} \oint_{|z|=\delta} \frac{f(z)}{z^{-n+1}} dz \right| \le \frac{1}{2\pi} \delta^{n-1} M \cdot 2\pi\delta \to 0 \text{ as } \delta \downarrow 0$$

and hence the Laurent series of f has zero principle part.

If $f(D'(z_0,\delta))$ is **not** dense in $\mathbb{C}$ for some $0 < \delta < \varepsilon$, then there exists $w \in \mathbb{C}$ and $\rho > 0$ such that $|f(z) - w| \ge \rho$ for all $z \in D'(z_0,\delta)$ and hence $g(z) := \frac{1}{f(z)-w}$ is bounded for z near z_0 and therefore we may make g analytic in a neighborhood of z_0 by setting $g(z_0) = \lim_{z\to z_0} g(z)$. Assuming f is not constant we know that g has a zero of order $n \in \mathbb{N}_0$ at z_0 and hence we may write

$$\frac{1}{f(z) - w} = g(z) = (z - z_0)^n \psi(z)$$

where ψ is analytic near z_0 with $\psi(z_0) \ne 0$. Thus it follows that

$$f(z) = w + \frac{1}{(z - z_0)^n \psi(z)}$$

showing that f has a pole of order n at z_0. So we have shown if f is not constant and $f(D'(z_0,\delta))$ is **not** dense in $\mathbb{C}$ for some $0 < \delta < \varepsilon$, then f has a pole or a removable singularity at z_0. Conversely if f has a pole at z_0, then $\lim_{z\to\infty} f(z_0) = \infty$ or if f has a removable singularity at z_0 then $\lim_{z\to\infty} f(z_0) = L \in \mathbb{C}$. In either case, $f(D'(z_0,\delta))$ is **not** dense in $\mathbb{C}$.

Thus there is only one possibility left to consider, i.e. where $f(D'(z_0,\delta))$ is dense in $\mathbb{C}$. This last possibility necessarily must correspond to the case where f has an essential singularity at z_0. This case may be distinguished by checking that $\lim_{z\to z_0} f(z)$ does not exists in $\mathbb{C}_\infty$. ■

Example 43.2. To illustrate item 3. of Theorem 43.1 above consider $f(z) = e^{1/z}$ which has an essential singularity at 0. Note if $0 \ne w = re^{i\theta} = e^{\ln r + i\theta}$, then $e^{1/z} = w = e^{\ln r + i\theta}$ iff

$$\frac{1}{z} = \ln r + i\theta + i2\pi k \text{ for some } k \in \mathbb{Z} \iff z = z_k := \frac{1}{\ln r + i\theta + i2\pi k}.$$

Since $z_k \to 0$ as $k \to \infty$ from which it follows $f(D'(0,\delta)) = \mathbb{C} \setminus \{0\}$ for all $\delta > 0$. This example illustrated Great Picard's Theorem.

Theorem 43.3 (Great Picard's Theorem). *If an analytic function f has an essential singularity at a point z_0, then on any punctured neighborhood of z_0, $f(z)$ takes on all possible complex values, with at most a single exception, infinitely often.*

43.2 The Theory of Partial Fractions

Let us give a complex variables interpretation of the theory behind partial fractions.

Example 43.4. If

$$f(z) = \frac{z}{(z-1)(z-3)^2}$$

then f has a pole of order 2 at 3 and so there exists $a, b \in \mathbb{C}$ so that

$$g(z) = f(z) - \left[\frac{a}{z-3} + \frac{b}{(z-3)^2}\right]$$

where $\frac{a}{z-3} + \frac{b}{(z-3)^2}$ is the principle part of f near 3. The function $g(z)$ has removable singularity at 3 and hence may be extended to an analytic on $\mathbb{C} \setminus$

$\{1\}$. This function satisfies, $\lim_{z\to\infty} g(z) = 0$ and has a pole of order 1 at 1. Following the same logic let $\frac{c}{z-1}$ be the principle part of g near 1, then

$$h(z) = g(z) - \frac{c}{z-1}$$

is analytic on $\mathbb{C}$ and moreover

$$\lim_{z\to\infty} h(z) = \lim_{z\to\infty}\left[g(z) - \frac{c}{z-1}\right] = 0.$$

It now follows that h is a bounded entire function and hence $h(z)$ is constant by Louiville's theorem. Since $\lim_{z\to\infty} h(z) = 0$, the constant value of h must be 0, i.e. $h(z) = 0$ for all $z \in \mathbb{C}$. Thus we have shown there exists $a, b, c \in \mathbb{C}$ such that

$$0 = g(z) - \frac{c}{z-1} = f(z) - \left[\frac{a}{z-3} + \frac{b}{(z-3)^2}\right] - \frac{c}{z-1},$$

i.e. the following partial fraction expansion holds;

$$\frac{z}{(z-1)(z-3)^2} = f(z) = \frac{a}{z-3} + \frac{b}{(z-3)^2} + \frac{c}{z-1}. \tag{43.1}$$

We can further find the coefficients using

$$c = res_{z=1}\frac{z}{(z-1)(z-3)^2} = \frac{1}{(1-3)^2} = \frac{1}{4}$$
$$a = res_{z=3}\frac{z}{(z-1)(z-3)^2} = \frac{1}{1!}\frac{d}{dz}|_{z=3}\frac{z}{z-1} = \frac{1}{1!}\cdot\frac{3-1-3}{(3-1)^2} = -\frac{1}{4}.$$

Lastly,

$$b = res_3\left[\frac{z}{(z-1)(z-3)^2}(z-3)\right] = \frac{3}{3-1} = \frac{3}{2}$$

and we have shown,

$$\frac{z}{(z-1)(z-3)^2} = \frac{1}{4}\frac{1}{z-1} + \frac{3}{2}\frac{1}{(z-3)^2} - \frac{1}{4}\frac{1}{z-3}.$$

The fact that the coefficients of the $\frac{1}{1-z}$ and $\frac{1}{z-3}$ terms sum to 0 is a consequence of the fact that $f(z) = O\left(\frac{1}{z^2}\right)$ as $z \to \infty$. This sort of statement holds more generally.

Theorem 43.5 (Integrals of rational functions). *For any rational function, $f(x) = p(x)/q(x)$ (where $p(x)$ and $q(x)$ are polynomials with possibly complex coefficients), one can always find an indefinite integral, $F(x) = \int f(x)\,dx$.*

Proof. By dividing q into p if necessary, there is not loss in generality assuming that $\deg p < \deg q$. To complete the proof, we decompose f into its partial fraction decomposition in Eq. (**??**) which reduces the problem to finding anti-derivatives for $(x-w)^{-k}$ for $k \in \mathbb{N}$ which is easy to do, namely

$$\int (x-w)^{-k}\,dx = \begin{cases} \frac{1}{1-k}(x-w)^{-k+1} & \text{if } k > 1 \\ \text{Log}(x-w) & \text{if } k = 1 \end{cases}$$

if $k = 1$ provided w is not real. If w is real we take $\int (x-w)^{-1}\,dx = \ln|x-w| + C$ as usual. ■

Example 43.6. If

$$f(x) = \frac{1}{1+x^2} = \frac{1}{2i}\frac{1}{x-i} - \frac{1}{2i}\frac{1}{x+i} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right)$$

and hence

$$\int f(x)\,dx = \frac{1}{2i}\left[\text{Log}(x-i) - \text{Log}(x+i)\right] + C.$$

Referring to Figure 43.1, we see that

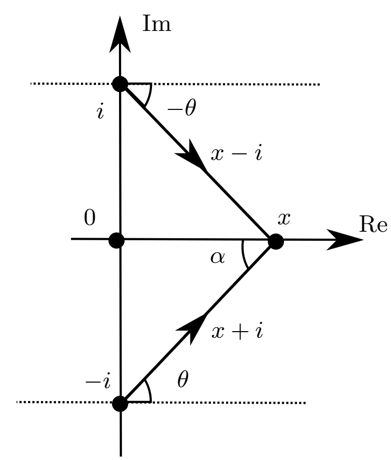

Fig. 43.1. The geometry involved with computing the logarithms.

$$\begin{aligned}\text{Log}(x-i) - \text{Log}(x+i) &= [\ln|x-i| - i\theta] - [\ln|x+i| + i\theta] \\ &= -2i\theta = -2i(\pi/2 - \alpha) = 2i\left(\tan^{-1}(x) - \pi/2\right)\end{aligned}$$

and hence

$$\int f(x)\,dx = \frac{1}{2i} 2i\left(\tan^{-1}(x) - \pi/2\right) + C = \tan^{-1}(x) + C$$

as is we all know to be the case.

Remark 43.7. If $p(x)$ and $q(x)$ are polynomials with complex coefficients such $q(x) \neq 0$ for all $x \in \mathbb{R}$ and $\deg q \geq \deg p + 2$, then

$$\frac{p(x)}{q(x)} = \sum_{z:q(z)=0} \left(res_z \frac{p}{q}\right) \cdot \frac{1}{x-z} + g(x)$$

where $g(x)$ is a linear combination of terms for the form $\left(\frac{1}{x-w}\right)^k$ with $k \in \{2,3,4,\dots\} = \mathbb{N} \setminus \{1\}$ and $w \in \mathbb{C} \setminus \mathbb{R}$. Since

$$\int_{-\infty}^{\infty} (x-w)^{-k}\,dx = \frac{1}{1-k}(x-w)^{-k+1}|_{-\infty}^{\infty} = 0 \text{ for } k \geq 2$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = \int_{-\infty}^{\infty} \left[\sum_{z:q(z)=0} \left(res_z \frac{p}{q}\right) \cdot \frac{1}{x-z}\right] dx.$$

As hinted at the end of Example 43.4 we know that

$$\sum_{z:q(z)=0} res_z \frac{p}{q} = 0,$$

a fact that also follows by computing residues at ∞ (see homework problem) or by showing

$$\oint_{|z|=R} \frac{p(z)}{q(z)} dz = O\left(\frac{1}{R}\right) \to 0 \text{ as } R \to \infty.$$

Thus with some work one shows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left[\sum_{z:q(z)=0} res_{z=0} \frac{p(z)}{q(z)} \frac{1}{x-z}\right] dx \\
&= \lim_{R\to\infty} \sum_{z:q(z)=0} res_z \frac{p}{q} \cdot \operatorname{Log}(x-z)|_{-R}^{R} \\
&= \sum_{\operatorname{Im} z>0} res_z \frac{p}{q} \cdot i\pi - i\pi \sum_{\operatorname{Im} z<0} res_z \frac{p}{q} \\
&= i2\pi \cdot \sum_{\operatorname{Im} z>0} res_z \frac{p}{q} = -i2\pi \cdot \sum_{\operatorname{Im} z<0} res_z \frac{p}{q}
\end{aligned}$$

which is a result we will again easily prove below. **However,** partial fractions is basically applicable in all situations but may be more work than is necessary in special cases of interest.