Mean value and maximum (minimum) principles (6/6/2018)

Corollary 68.1 (Mean value property). *Let $\Omega \subset_o \mathbb{C}$ and $f \in H(\Omega)$, then f satisfies the mean value property*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta})d\theta \tag{68.1}$$

which holds for all z_0 and $\rho \geq 0$ such that $\overline{D(z_0,\rho)} \subset \Omega$.

Proof. By Cauchy's integral formula and parametrizing $\partial D(z_0,\rho)$ as $z = z_0 + \rho e^{i\theta}$, we learn

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_{\partial D(z_0,\rho)} \frac{f(z)}{z - z_0}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+\rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+\rho e^{i\theta})d\theta. \end{aligned}$$

■

Theorem 68.2 (Mean Value Property for Harmonic Functions). *If $u : \overline{D(z_0,r)} \to \mathbb{R}$ be a continuous function which is harmonic on $D(z_0,r)$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right)d\theta = \frac{1}{2\pi i}\oint_{|z-z_0|=r} u(z)\frac{dz}{z}. \tag{68.2}$$

Proof. Let v be a harmonic conjugate to u (see Corollary 64.10) so that $f = u + iv$ is analytic on $D(z_0,r)$. For $0 < \rho < r$, we take the real part of Eq. (68.1) to find arrive at

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+\rho e^{i\theta})d\theta.$$

We then let $\rho \uparrow t$ to arrive at Eq. (68.2). ■

Theorem 68.3 (Maximum principle for Harmonic Functions). *Suppose that Ω is an open connected region and $u : \Omega \to \mathbb{R}$ is a harmonic function. If u has a local maximum (minimum) at some point $z_0 \in \Omega$, then u is constant.*

Proof. From Eq. (68.2) and the given assumptions,

$$0 = u(z_0) - \frac{1}{2\pi}\int_0^{2\pi} u\left(z_0 + re^{i\theta}\right)d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left[u(z_0) - u\left(z_0+re^{i\theta}\right)\right]d\theta$$

where for r sufficiently small, $u(z_0) - u\left(z_0+re^{i\theta}\right) \geq 0$ and hence we must have

$$u(z_0) - u\left(z_0+re^{i\theta}\right) = 0 \text{ for all } \theta \text{ and } r \text{ small.}$$

This shows that u is constant near z_0. Given another point $z \in \Omega$, we choose an analytic function, f defined on a simply connected region containing z_0 and z such that $\operatorname{Re} f = u$. By the open mapping theorem and analytioc continuation methods it follows that f is constant on the this region and hence $u(z) = \operatorname{Re} f(z) = \operatorname{Re} f(z_0) = u(z_0)$. ■

Corollary 68.4 (Harmonic function maximum principle). *Let Ω be a bounded region and $u \in C(\overline{\Omega},\mathbb{R})$ such that $\Delta u(z) = 0$ for $z \in \Omega$. Then for all $z \in \Omega$,*

$$\min_{z\in\partial\Omega} u(z) \leq u(z) \leq \max_{z\in\partial\Omega} u(z).$$

Furthermore if there exists $z_0 \in \Omega$ such that $u(z_0)$ is either a minimum or a maximum, then u is constant.

Corollary 68.5 (Dirichlet problem uniqueness). *Let Ω be a bounded region and $g \in C(\partial\Omega,\mathbb{R})$ be a given function, then there is at most one function $u \in C(\overline{\Omega},\mathbb{R})$ such that $\Delta u(z) = 0$ for $z \in \Omega$ and $u = g$ on $\partial\Omega$.*

Proof. If there was another function w then $v = u - w$ would solve $\Delta v = 0$ with $v = 0$ on $\partial\Omega$ and then by Corollary 68.4 it follows that $v = 0$, i.e. $u = w$. ■

Solving the Dirichlet problems on D

Theorem 69.1 (Change of Variables). *Let C be a finite length contour h be an analytic function on a domain, D, such that $C \subset D$, and $f : h \circ C \to \mathbb{C}$ be a continuous function, then,*

$$\int_C f(h(z))\, h'(z)\, dz = \int_{h \circ C} f(w)\, dw.$$

In short, if we let $w = h(z)$, then $dw = h'(z)\, dz$.

Proof. Let $[a, b] \ni t \to z(t)$ be a parametrization of C and so $w(t) := h(z(t))$ is a parameterization of $h(C)$. Therefore,

$$\int_{h \circ C} f(w)\, dw = \int_a^b f(w(t))\, \dot{w}(t)\, dt = \int_a^b f(h(z(t)))\, h'(z(t))\, \dot{z}(t)\, dt$$
$$= \int_C f(h(z))\, h'(z)\, dz.$$

■

Notation 69.2 *For the rest of this section, let $D = D(0, 1)$ be the open unit disk centered at $0 \in \mathbb{C}$ and $\bar{D}$ be the closed unit disk.*

Remark 69.3. If $h(z) = f(z)/g(z)$ where f, g are analytic functions near z_0, $g(z_0) \neq 0 \neq f(z_0)$, then

$$\frac{h'(z)}{h(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}.$$

This is easily verified since, by the quotient rule,

$$\frac{h'}{h} = \frac{f'g - fg'}{g^2} \cdot \frac{1}{h} = \frac{f'g - fg'}{g^2} \frac{g}{f} = \frac{f'}{f} - \frac{g'}{g}.$$

Lemma 69.4. *If, for $\xi \in D$, φ_ξ is the LFT (Möbius transform),*

$$\varphi_\xi(z) := \frac{z - \xi}{1 - \bar{\xi} z}, \tag{69.1}$$

then $\varphi_\xi : \bar{D} \to \bar{D}$ is a homeomorphism, $\varphi_\xi : D \to D$ is a conformal, $\varphi_\xi(\partial D) = \partial D$, and for $|z| = 1$,

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\bar{\xi}}{1 - \bar{\xi} z} \tag{69.2}$$

$$= \frac{1 - |\xi|^2}{|z - \xi|^2} \frac{1}{z}. \tag{69.3}$$

Moreover, $\theta \to \varphi_\xi\left(e^{i\theta}\right)$ traverses the ∂D in the counter-clockwise direction.

Proof. The stated mapping properties of φ_ξ have already been proved in Theorem 59.7 above. From Remark 69.3,

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\bar{\xi}}{1 - \bar{\xi} z} \text{ for } |z| = 1 \text{ and } |\xi| < 1, \tag{69.4}$$

which proves Eq. (69.2). If $|z| = 1$ (i.e. $z\bar{z} = 1$), we have

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\bar{\xi}}{1 - \bar{\xi} z} = \frac{1 - \bar{\xi} z + \bar{\xi}(z - \xi)}{(z - \xi)\left(1 - \bar{\xi} z\right)}$$
$$= \frac{1}{z\bar{z}} \frac{1 - |\xi|^2}{(z - \xi)\left(1 - \bar{\xi} z\right)} = \frac{1}{z} \frac{1 - |\xi|^2}{(1 - \bar{z}\xi)\left(1 - \bar{\xi} z\right)}$$
$$= \frac{1}{z} \frac{1 - |\xi|^2}{|1 - \bar{z}\xi|^2} = \frac{1}{z} \frac{1 - |\xi|^2}{|z - \xi|^2}$$

from which we deduce Eq. (69.3).

The last assertion is a consequence of the identity,

$$\frac{d}{d\theta} \varphi_\xi\left(e^{i\theta}\right) = ie^{i\theta} \varphi_\xi'\left(e^{i\theta}\right) = ie^{i\theta} \frac{1}{e^{i\theta}} \frac{1 - |\xi|^2}{\left|e^{i\theta} - \xi\right|^2} \varphi_\xi\left(e^{i\theta}\right)$$
$$= i \frac{1 - |\xi|^2}{\left|e^{i\theta} - \xi\right|^2} \varphi_\xi\left(e^{i\theta}\right) = i\gamma \varphi_\xi\left(e^{i\theta}\right)$$

where $\gamma > 0$ and therefore $\frac{d}{d\theta} \varphi_\xi\left(e^{i\theta}\right)$ is a tangent vector to ∂D at $\varphi_\xi\left(e^{i\theta}\right)$ which points in the counter-clockwise direction. ■

Proposition 69.5 (Mean value property). *If $u : \bar{D} \to \mathbb{R}$ is continuous function which is harmonic inside of D, then*

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u\left(e^{i\theta}\right) d\theta = \frac{1}{2\pi i} \oint_{|w|=1} u(w) \frac{dw}{w}. \tag{69.5}$$

Proof. Let v be the Harmonic conjugate to u so that $f = u + iv$ is analytic on D. Then by the Cauchy integral formula we know that

$$f(0) = \frac{1}{2\pi i}\oint_{|w|=r}\frac{f(w)}{w}dw = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(re^{i\theta}\right)d\theta \text{ for all } 0<r<1.$$

Taking the real part of this equation then shows

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right)d\theta$$

and then letting $r \uparrow 1$ proves Eq. (69.5). ■

Corollary 69.6 (Representation formula). *If $u : \bar{D} \to \mathbb{R}$ is continuous function which is harmonic inside of D, then, for $|\xi| < 1$,*

$$\begin{aligned} u(\xi) &= \frac{1}{2\pi i}\oint_{\partial D} u\left(\varphi_{-\xi}(w)\right)\frac{dw}{w} &(69.6)\\ &= \frac{1}{2\pi i}\oint_{\partial D} u(z)\frac{\varphi_\xi'(z)}{\varphi_\xi(z)}dz &(69.7)\\ &= \frac{1}{2\pi i}\oint_{\partial D} u(z)\left(\frac{1}{z-\xi}+\frac{\bar\xi}{1-\bar\xi z}\right)dz &(69.8)\\ &= \frac{1}{2\pi i}\oint_{\partial D} u(z)\frac{1-|\xi|^2}{|z-\xi|^2}\frac{dz}{z} &(69.9)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(e^{i\theta}\right)\frac{1-|\xi|^2}{\left|e^{i\theta}-\xi\right|^2}d\theta. &(69.10)\end{aligned}$$

Proof. Let $\xi \in D$ and $\varphi_{-\xi}$ be the LFT as in Eq. (69.1). As we have seen some time ago, $\varphi_{-\xi} : \bar{D} \to \bar{D}$ is a homeomorphism, $\varphi_{-\xi} : \partial D \to \partial D$, and $\varphi_{-\xi} : D \to D$ is conformal. Hence $u \circ \varphi_{-\xi} = \operatorname{Re} f \circ \varphi_{-\xi}$ is still harmonic and is still continuous on $\bar{D}$. Therefore we may apply the mean value property in Eq. (69.5) with u replaced by $u \circ \varphi_{-\xi}$ shows gives Eq. (69.6) for all $\xi \in D$. Using Theorem 39.4, we make the change of variables, $w = \varphi_\xi(z)$, to find with the aid of Lemma[1] 69.4 that Eq. (69.6) may be rewritten as in Eqs. (69.7–69.9) while Eq. (69.10) then follows by definition of the contour integral around ∂D in the counter-clockwise orientation.

$$\begin{aligned} u(\xi) = u\circ\varphi_{-\xi}(0) &= \frac{1}{2\pi i}\oint_{\partial D} u(z)\frac{1-|\xi|^2}{|z-\xi|^2}\frac{dz}{z}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u(z)\frac{1-|\xi|^2}{\left|e^{i\theta}-\xi\right|^2}d\theta\end{aligned}$$

[1] We also use $\varphi_{-\xi}(\partial D) = \partial D = \varphi_\xi(\partial D)$ and preserves orientation on the boundary which can be verified directly as we did in Lemma 69.4 or deduced from the fact that $\varphi_{-\xi}$ is conformal and takes D to D.

which is Eq. (69.10). ■

Theorem 69.7 (Solving the Dirichlet Problem). *For $g \in C(\partial D, \mathbb{R})$, let*

$$u_g(\xi) := \frac{1}{2\pi}\int_{-\pi}^{\pi} g\left(e^{i\theta}\right)\frac{1-|\xi|^2}{\left|e^{i\theta}-\xi\right|^2}d\theta \text{ for } |\xi|<1 \tag{69.11}$$

If we extend u_g to the ∂D by setting, $u_g = g$ on ∂D, then $u_g : \bar{D} \to \mathbb{R}$ is continuous, $u_g = g$ on ∂D, and $\Delta u_g = 0$ in D.

Proof. As in Corollary 69.6, we may also write $u_g(\xi)$ as

$$\begin{aligned} u_g(\xi) &= \frac{1}{2\pi i}\oint_{\partial D} g(z)\left(\frac{1}{z-\xi}+\frac{\bar\xi}{1-\bar\xi z}\right)dz &(69.12)\\ &= \frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{1}{z-\xi}dz + \frac{1}{2\pi i}\oint_{\partial D} g(z)\frac{\bar\xi}{1-\bar\xi z}dz\end{aligned}$$

from which it follows that u_g is the sum of a holomorphic and anti-holmorphic function and therefore u_g is harmonic.

Similarly as in Corollary 69.6 we can also write $u_g(\xi)$ as

$$u_g(\xi) = \frac{1}{2\pi i}\oint_{\partial D} g\left(\varphi_{-\xi}(w)\right)\frac{dw}{w}.$$

We now let $\xi = rv$ with $0 \le r < 1$ and $|v| = 1$ and then compute,

$$\varphi_{-\xi}(w) = \varphi_{-rv}(w) = \frac{w+rv}{1+r\bar{v}w} = v\cdot\bar{v}\frac{w+rv}{1+r\bar{v}w} = v\cdot\frac{\bar{v}w+r}{1+r\bar{v}w}$$

and so

$$u_g(rv) = \frac{1}{2\pi i}\oint_{\partial D} g\left(v\cdot\frac{\bar{v}w+r}{1+r\bar{v}w}\right)\frac{dw}{w} = \frac{1}{2\pi i}\oint_{\partial D} g\left(v\cdot\frac{z+r}{1+rz}\right)\frac{dz}{z}$$

wherein we made the change of variables $z = \bar{v}w$ in the last equality. Since

$$\lim_{r\uparrow 1}\frac{z+r}{1+rz} = \begin{cases} 1 & \text{if } z \neq -1\\ -1 & \text{if } z = -1\end{cases}$$

we may conclude (by DCT) that

$$\lim_{r\uparrow 1} u_g(rv) = \frac{1}{2\pi i}\oint_{\partial D} g(v)\frac{dz}{z} = g(v).$$

In fact if we let $\delta_g(\eta) := \max_{|v|=1}|g(v\eta) - g(v)|$ which goes to zero as $\eta \to 1$ by uniform continuity of g on ∂D, we may further conclude

$$\max_{|v|=1} |u_g(rv) - g(v)| = \max_{|v|=1} \left| \frac{1}{2\pi i} \oint_{\partial D} \left[g\left(v \cdot \frac{z+r}{1+rz} \right) - g(v) \right] \frac{dz}{z} \right|$$
$$\leq \oint_{\partial D} \delta_g \left(\frac{z+r}{1+rz} \right) |dz| \to 0 \text{ as } r \uparrow 1.$$

This last assertion easily implies shows that u_g, as defined, is in fact continuous on $\bar{D}$. ■

Definition 69.8 (Poisson Kernel). *For* $0 \leq r < 1$ *and* $\theta \in \mathbb{R}$, *let*

$$p_r(\theta) := \frac{1-r^2}{1+r^2-2r\cos(\theta)}$$

which we referred to as the ***Poisson kernel.***

Corollary 69.9 (Poisson Integral Formula). *If* $g \in C(\partial D, \mathbb{R})$, $0 \leq r < 1$, *and* $\alpha \in \mathbb{R}$, *then*

$$u_g\left(re^{i\alpha}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\alpha - \theta)\, g\left(e^{i\theta}\right) d\theta. \tag{69.13}$$

Proof. If we write $\xi = re^{i\alpha}$, then

$$\frac{1-|\xi|^2}{|e^{i\theta}-\xi|^2} = \frac{1-\left|re^{i\alpha}\right|^2}{\left|e^{i\theta}-re^{i\alpha}\right|^2} = \frac{1-r^2}{1+r^2-2r\cos(\alpha-\theta)}$$
$$= \frac{1-r^2}{1+r^2-2r\cos(\theta)}$$

and so Eq. (69.11) is equivalent to Eq. (69.13). ■

Theorem 69.10 (Fourier series representation). *The function* u_g *in Theorem 69.7 has the "Fourier series representation,"*

$$u_g\left(re^{i\alpha}\right) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{g}(n)\, e^{in\alpha} \tag{69.14}$$

where

$$\hat{g}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g\left(e^{i\theta}\right) e^{-in\theta} d\theta. \tag{69.15}$$

Proof. We start with the expression,

$$u_g(\xi) = \frac{1}{2\pi i} \oint_{\partial D} g(z) \left(\frac{z}{z-\xi} + \frac{\bar{\xi}z}{1-\bar{\xi}z} \right) \frac{dz}{z} \text{ for } |\xi| < 1.$$

By geometric series considerations,

$$\frac{z}{z-\xi} = \frac{1}{1-\xi/z} = \sum_{n=0}^{\infty} \left(\frac{\xi}{z} \right)^n = \sum_{n=0}^{\infty} \xi^n z^{-n},$$

$$\frac{\bar{\xi}z}{1-\bar{\xi}z} = \bar{\xi}z \sum_{n=0}^{\infty} \left[\bar{\xi}z\right]^n = \sum_{n=1}^{\infty} \bar{\xi}^n z^n,$$

and hence,

$$\frac{z}{z-\xi} + \frac{\bar{\xi}z}{1-\bar{\xi}z} = 1 + \sum_{n=1}^{\infty} \left[\xi^n z^{-n} + \bar{\xi}^n z^n \right].$$

Letting $z = e^{i\theta}$, this expression becomes,

$$\frac{z}{z-\xi} + \frac{\bar{\xi}z}{1-\bar{\xi}z} = 1 + \sum_{n=1}^{\infty} \left[\xi^n e^{-i\theta n} + \bar{\xi}^n e^{i\theta n} \right]$$

and so

$$u_g(\xi) = \frac{1}{2\pi} \oint_{\partial D} g\left(e^{i\theta}\right) \left(1 + \sum_{n=1}^{\infty} \left[\xi^n e^{-i\theta n} + \bar{\xi}^n e^{i\theta n} \right] \right) d\theta$$
$$= \hat{g}(0) + \sum_{n=1}^{\infty} \left[\xi^n \hat{g}(n) + \bar{\xi}^n \hat{g}(-n) \right].$$

Now writing $\xi = re^{i\alpha}$ we further find,

$$u\left(re^{i\alpha}\right) = \hat{g}(0) + \sum_{n=1}^{\infty} r^n \left[\hat{g}(n)\, e^{in\alpha} + \hat{g}(-n)\, e^{-in\alpha} \right] = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{g}(n)\, e^{in\alpha}.$$

■

Theorem 69.11 (Fourier's Theorem). *If* $g : \partial D \to \mathbb{C}$ *is a function such that* $\alpha \to g\left(e^{i\alpha}\right)$ *is continuously differentiable, then*

$$g\left(e^{i\alpha}\right) = \sum_{n=-\infty}^{\infty} \hat{g}(n)\, e^{in\alpha} \text{ for all } \alpha \in \mathbb{R}, \tag{69.16}$$

where the sum is absolutely (hence uniformly convergent). As before, $\hat{g}(n)$ *is as was defined in Eq. (69.15).*

Proof. By an integration by parts argument one and Bessel's inequality one shows

$$\sum_{n=-\infty}^{\infty} n^2 |\hat{g}(n)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{d}{d\alpha} g\left(e^{i\alpha}\right) \right|^2 d\alpha < \infty.$$

This inequality along with the Cauchy-Schwarz inequality, then shows

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}} |\hat{g}(n)| &= \sum_{n\in\mathbb{Z}\setminus\{0\}} |\hat{g}(n)|\,|n| \cdot \frac{1}{|n|} \\ &\leq \left(\sum_{n=-\infty}^{\infty} n^2 |\hat{g}(n)|^2\right)^{1/2} \cdot \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2}\right)^{1/2} < \infty.\end{aligned}$$

This shows the infinite sum in Eq. (69.16) is absolutely (hence uniformly convergent). Thus we pass to the limit as $r \uparrow 1$ in Eq. (69.14) in order to arrive at the equality in Eq. (69.16). ■