Mean value and maximum (minimum) principles (6/6/2018)

Corollary 68.1 (Mean value property). Let \( \Omega \subset \mathbb{C} \) and \( f \in H(\Omega) \), then \( f \) satisfies the mean value property

\[
f(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta \tag{68.1}
\]

which holds for all \( z_0 \) and \( \rho \geq 0 \) such that \( D(\Omega, \rho) \subset \Omega \).

**Proof.** By Cauchy’s integral formula and parametrizing \( \partial D(z_0, \rho) \) as \( z = z_0 + \rho e^{i\theta} \), we learn

\[
f(z_0) = \frac{1}{2\pi} \int_{\partial D(z_0, \rho)} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{0}^{2\pi} \frac{f(z_0 + \rho e^{i\theta})}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} f(z_0 + \rho e^{i\theta}) d\theta.
\]

Theorem 68.2 (Mean Value Property for Harmonic Functions). If \( u : D(z_0, r) \to \mathbb{R} \) be a continuous function which is harmonic on \( D(z_0, r) \), then

\[
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{|z - z_0| = r} u(z) \frac{dz}{z} \tag{68.2}
\]

**Proof.** Let \( v \) be a harmonic conjugate to \( u \) (see Corollary 64.10) so that \( f = u + iv \) is analytic on \( D(z_0, r) \). For \( 0 < \rho < r \), we take the real part of Eq. \( 68.1 \) to find arrive at

\[
u(z_0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + se^{i\theta}) d\theta.
\]

We then let \( \rho \uparrow r \) to arrive at Eq. \( 68.2 \).

Theorem 68.3 (Maximum principle for Harmonic Functions). Suppose that \( \Omega \) is an open connected region and \( u : \Omega \to \mathbb{R} \) is a harmonic function. If \( u \) has a local maximum (minimum) at some point \( z_0 \in \Omega \), then \( u \) is constant.

**Proof.** From Eq. \( 68.2 \) and the given assumptions,

\[
0 = u(z_0) - \frac{1}{2\pi} \int_{0}^{2\pi} u(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} [u(z_0) - u(z_0 + re^{i\theta})] d\theta
\]

where for \( r \) sufficiently small, \( u(z_0) - u(z_0 + re^{i\theta}) \geq 0 \) and hence we must have

\[
u(z_0) - u(z_0 + re^{i\theta}) = 0\, \text{for all } \theta \text{ and } r \text{ small.}
\]

This shows that \( u \) is constant near \( z_0 \). Given another point \( z \in \Omega \), we choose an analytic function, \( f \) defined on a simply connected region containing \( z_0 \) and \( z \) such that \( \text{Re } f = u \). By the open mapping theorem and analytic continuation methods it follows that \( f \) is constant on the this region and hence \( u(z) = \text{Re } f(z) = \text{Re } f(z_0) = u(z_0) \).

Corollary 68.4 (Harmonic function maximum principle). Let \( \Omega \) be a bounded region and \( u \in C(\overline{\Omega}, \mathbb{R}) \) such that \( \Delta u(z) = 0 \) for \( z \in \Omega \). Then for all \( z \in \Omega \),

\[
\min_{z \in \partial \Omega} u(z) \leq u(z) \leq \max_{z \in \partial \Omega} u(z).
\]

Furthermore if there exists \( z_0 \in \Omega \) such that \( u(z_0) \) is either a minimum or a maximum, then \( u \) is constant.

Corollary 68.5 (Dirichlet problem uniqueness). Let \( \Omega \) be a bounded region and \( g \in C(\partial \Omega, \mathbb{R}) \) be a given function, then there is at most one function \( u \in C(\overline{\Omega}, \mathbb{R}) \) such that \( \Delta u(z) = 0 \) for \( z \in \Omega \) and \( u = g \) on \( \partial \Omega \).

**Proof.** If there was another function \( w \) then \( v = u - w \) would solve \( \Delta v = 0 \) with \( v = 0 \) on \( \partial \Omega \) and then by Corollary 68.4 it follows that \( v = 0 \), i.e. \( u = w \).
Solving the Dirichlet problems on $D$

**Theorem 69.1 (Change of Variables).** Let $C$ be a finite length contour $h$ be an analytic function on a domain, $D$, such that $C \subset D$, and $f : h \circ C \to \mathbb{C}$ be a continuous function, then,

$$\int_C f(h(z)) h'(z) \, dz = \int_{h\circ C} f(w) \, dw.$$ 

In short, if we let $w = h(z)$, then $dw = h'(z) \, dz$.

**Proof.** Let $[a, b] \ni t \to z(t)$ be a parametrization of $C$ and so $w(t) := h(z(t))$ is a parameterization of $h(C)$. Therefore,

$$\int_{h\circ C} f(w) \, dw = \int_a^b f(w(t)) \dot{w}(t) \, dt = \int_a^b f(h(z(t))) h'(z(t)) \dot{z}(t) \, dt = \int_C f(h(z)) h'(z) \, dz.$$ 

$\blacksquare$

**Notation 69.2** For the rest of this section, let $D = D(0, 1)$ be the open unit disk centered at $0 \in \mathbb{C}$ and $\bar{D}$ be the closed unit disk.

**Remark 69.3.** If $h(z) = f(z)/g(z)$ where $f, g$ are analytic functions near $z_0$, $g(z_0) \neq 0 \neq f(z_0)$, then

$$h'(z) = \frac{f'(z) g(z) - f(z) g'(z)}{g(z)^2}.$$ 

This is easily verified since, by the quotient rule,

$$\frac{h'}{h} = \frac{f'g - fg' g}{g^2} = \frac{f'g - fg' f}{f^2} = \frac{f'}{f} - \frac{g'}{g}.$$ 

**Lemma 69.4.** If, for $\xi \in D$, $\varphi_\xi$ is the LFT (Möbius transform),

$$\varphi_\xi(z) := \frac{z - \xi}{1 - \xi z},$$

then $\varphi_\xi : \bar{D} \to \bar{D}$ is a homeomorphism, $\varphi_\xi : D \to D$ is a conformal, $\varphi_\xi(\partial D) = \partial D$, and for $|z| = 1$,

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\xi}{1 - \xi z} = 1 - |\xi|^2 \frac{1}{z - \xi z}. \quad (69.2)$$

Moreover, $\theta \to \varphi_\xi(e^{i\theta})$ traverses the $\partial D$ in the counter-clockwise direction.

**Proof.** The stated mapping properties of $\varphi_\xi$ have already been proved in Theorem 59.7 above. From Remark 69.3,

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\xi}{1 - \xi z}$$

for $|z| = 1$ and $|\xi| < 1$, \hfill (69.4)

which proves Eq. (69.2). If $|z| = 1$ (i.e. $z\bar{z} = 1$), we have

$$\frac{\varphi_\xi'(z)}{\varphi_\xi(z)} = \frac{1}{z - \xi} + \frac{\xi}{1 - \xi z} = \frac{1 - \xi z + \xi (z - \xi)}{(z - \xi)(1 - \xi z)} = \frac{1}{\overline{z} (z - \xi)} = \frac{1}{z (1 - \xi) (1 - \xi z)}$$

from which we deduce Eq. (69.3).

The last assertion is a consequence of the identity,

$$\frac{d}{d\theta} \varphi_\xi(e^{i\theta}) = i e^{i\theta} \frac{\varphi_\xi'(e^{i\theta})}{e^{i\theta}} = i e^{i\theta} \frac{1 - |\xi|^2}{|e^{i\theta} - \xi|^2} \varphi_\xi(e^{i\theta})$$

$$= i \frac{1 - |\xi|^2}{|e^{i\theta} - \xi|^2} \varphi_\xi(e^{i\theta}) = i \gamma \varphi_\xi(e^{i\theta})$$

where $\gamma > 0$ and therefore $\frac{d}{d\theta} \varphi_\xi(e^{i\theta})$ is a tangent vector to $\partial D$ at $\varphi_\xi(e^{i\theta})$ which points in the counter-clockwise direction. $\blacksquare$

**Proposition 69.5 (Mean value property).** If $u : \bar{D} \to \mathbb{R}$ is continuous function which is harmonic inside of $D$, then

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^\pi u(e^{i\theta}) \, d\theta = \frac{1}{2\pi i} \oint_{|w|=1} u(w) \frac{dw}{w}. \quad (69.5)$$
Corollary 69.6 (Representation formula). If \( u : \bar{D} \rightarrow \mathbb{R} \) is continuous function which is harmonic inside of \( D \), then, for \( |\xi| < 1 \),

\[
 u(\xi) = \frac{1}{2\pi i} \oint_{\partial D} u(\varphi_{-\xi}(w)) \frac{dw}{w}
\]

(69.6)

\[
= \frac{1}{2\pi i} \oint_{\partial D} u(z) \frac{\varphi'_z(z)}{\varphi(z)} dz
\]

(69.7)

\[
= \frac{1}{2\pi i} \int_{\partial D} u(z) \left( \frac{1}{z - \xi} + \frac{\bar{\xi}}{1 - \bar{\xi}z} \right) dz
\]

(69.8)

\[
= \frac{1}{2\pi i} \oint_{\partial D} u(z) \frac{1 - |\xi|^2}{|z - \xi|^2} \frac{dz}{z}
\]

(69.9)

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) \frac{1 - |\xi|^2}{|e^{i\theta} - \xi|^2} d\theta.
\]

(69.10)

**Proof.** Let \( v \) be the Harmonic conjugate to \( u \) so that \( f = u + iv \) is analytic on \( D \). Then by the Cauchy integral formula we know that

\[
f(0) = \frac{1}{2\pi i} \oint_{|w|=r} \frac{f(w)}{w} dw = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) d\theta \text{ for } 0 < r < 1.
\]

Taking the real part of this equation then shows

\[
u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta
\]

and then letting \( r \uparrow 1 \) proves Eq. (69.5).

Theorem 69.7 (Solving the Dirichlet Problem). For \( g \in C(\partial D, \mathbb{R}) \), let

\[
u_g(\xi) := \frac{1}{2\pi} \int_{\partial D} g \left( e^{i\theta} \right) \frac{1 - |\xi|^2}{|e^{i\theta} - \xi|^2} d\theta \text{ for } |\xi| < 1
\]

(69.11)

If we extend \( u_g \) to the \( \partial D \) by setting, \( u_g = g \) on \( \partial D \), then \( u_g : D \rightarrow \mathbb{R} \) is continuous, \( u_g = g \) on \( \partial D \), and \( \Delta u_g = 0 \) in \( D \).

**Proof.** As in Corollary 69.6, we may also write \( u_g(\xi) \) as

\[
u_g(\xi) = \frac{1}{2\pi i} \oint_{\partial D} g(z) \left( \frac{1}{z - \xi} + \frac{\bar{\xi}}{1 - \bar{\xi}z} \right) dz
\]

(69.12)

from which it follows that \( u_g \) is the sum of a holomorphic and anti-holomorphic function and therefore \( u_g \) is harmonic.

Similarly as in Corollary 69.6, we also write \( u_g(\xi) \) as

\[
u_g(\xi) = \frac{1}{2\pi i} \oint_{\partial D} g(\varphi_{-\xi}(w)) \frac{dw}{w}.
\]

We now let \( \xi = rv \) with \( 0 \leq r < 1 \) and \( |v| = 1 \) and then compute,

\[
\varphi_{-\xi}(w) = \varphi_{-rv}(w) = \frac{w + rv}{1 + rvw} = v \cdot \bar{v} \frac{w + rv}{1 + rvw} = v \cdot \bar{v} w + r
\]

and so

\[
u_g(rv) = \frac{1}{2\pi i} \oint_{\partial D} g \left( v \cdot \bar{v} \frac{w + rv}{1 + rvw} \right) \frac{dw}{w} = \frac{1}{2\pi i} \oint_{\partial D} g \left( v \cdot \bar{v} \frac{z + r}{1 + rz} \right) \frac{dz}{z}
\]

wherein we made the change of variables \( z = \bar{v} w \) in the last equality. Since

\[
l \lim_{r \uparrow 1} \frac{z + r}{1 + rz} = \begin{cases} 1 & \text{if } z \neq -1 \\ -1 & \text{if } z = -1 \end{cases}
\]

we may conclude (by DCT) that

\[
l \lim_{r \uparrow 1} \frac{z + r}{1 + rz} = \frac{1}{2\pi i} \oint_{\partial D} g(v) \frac{dz}{z} = g(v).
\]

In fact if we let \( \delta_\eta := \max_{|v|=1} |g(v)| - g(v)| \) which goes to zero as \( \eta \to 1 \) by uniform continuity of \( g \) on \( \partial D \), we may further conclude
\[
\max_{|v|=1} |u_g(rv) - g(v)| = \max_{|v|=1} \left| \frac{1}{2\pi i} \oint_{\partial D} \left[ g\left( v \cdot \frac{z + r}{1 + rz} \right) - g(v) \right] \frac{dz}{z} \right|
\]
\[
\leq \oint_{\partial D} \delta_g \left( \frac{z + r}{1 + rz} \right) |dz| \to 0 \text{ as } r \uparrow 1.
\]

This last assertion easily implies shows that \(u_g\), as defined, is in fact continuous on \(D\). □

**Definition 69.8 (Poisson Kernel).** For \(0 \leq r < 1\) and \(\theta \in \mathbb{R}\), let

\[
p_r(\theta) := \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta)}
\]

which we referred to as the Poisson kernel.

**Corollary 69.9 (Poisson Integral Formula).** If \(g \in C(\partial D, \mathbb{R})\), \(0 \leq r < 1\), and \(\alpha \in \mathbb{R}\), then

\[
u_g(re^{i\alpha}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\alpha - \theta) g(e^{i\theta}) d\theta.
\] (69.13)

**Proof.** If we write \(\xi = re^{i\alpha}\), then

\[
\frac{1 - |\xi|^2}{|e^{i\theta} - \xi|^2} = \frac{1 - |re^{i\alpha}|^2}{|e^{i\theta} - re^{i\alpha}|^2} = \frac{1 - r^2}{1 + r^2 - 2r \cos(\alpha - \theta)} = \frac{1 - r^2}{1 + r^2 - 2r \cos(\theta)}
\]

and so Eq. (69.11) is equivalent to Eq. (69.13). □

**Theorem 69.10 (Fourier series representation).** The function \(u_g\) in Theorem 69.7 has the “Fourier series representation,”

\[
u_g(re^{i\alpha}) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{g}(n) e^{in\alpha}
\] (69.14)

where

\[
\hat{g}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) e^{-in\theta} d\theta.
\] (69.15)

**Proof.** We start with the expression,

\[
u_g(\xi) = \frac{1}{2\pi i} \oint_{\partial D} g(z) \left( \frac{z}{z - \xi} + \frac{\xi}{1 - \xi} \right) \frac{dz}{z} \text{ for } |\xi| < 1.
\]

By geometric series considerations,

\[
z - \xi = \frac{1}{1 - \xi/z} = \sum_{n=0}^{\infty} \left( \frac{\xi}{z} \right)^n = \sum_{n=0}^{\infty} \xi^n z^{-n},
\]

\[
\frac{\xi z}{1 - \xi z} = \xi z \sum_{n=0}^{\infty} [\xi z]^n = \sum_{n=1}^{\infty} \xi^n z^n,
\]

and hence,

\[
z - \xi + \frac{\xi z}{1 - \xi z} = 1 + \sum_{n=1}^{\infty} [\xi^n z^{n-1} + \xi^n z^n].
\]

Letting \(z = e^{i\theta}\), this expression becomes,

\[
z - \xi + \frac{\xi z}{1 - \xi z} = 1 + \sum_{n=1}^{\infty} [\xi^n e^{-in\theta} + \xi^n e^{in\theta}]
\]

and so

\[
u_g(\xi) = \frac{1}{2\pi} \oint_{\partial D} g(e^{i\theta}) \left( 1 + \sum_{n=1}^{\infty} [\xi^n e^{-in\theta} + \xi^n e^{in\theta}] \right) d\theta
\]

\[= \hat{g}(0) + \sum_{n=1}^{\infty} \left[ \xi^n \hat{g}(n) + \xi^{-n} \hat{g}(-n) \right].
\]

Now writing \(\xi = re^{i\alpha}\) we further find,

\[
u_g(re^{i\alpha}) = \hat{g}(0) + \sum_{n=1}^{\infty} r^{|n|} \hat{g}(n) e^{in\alpha} + \hat{g}(-n) e^{-in\alpha} = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{g}(n) e^{in\alpha}.
\] □

**Theorem 69.11 (Fourier’s Theorem).** If \(g : \partial D \to \mathbb{C}\) is a function such that \(\alpha \to g(e^{i\alpha})\) is continuously differentiable, then

\[
g(e^{i\alpha}) = \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{in\alpha} \text{ for all } \alpha \in \mathbb{R},
\] (69.16)

where the sum is absolutely (hence uniformly convergent). As before, \(\hat{g}(n)\) is as was defined in Eq. (69.15).

**Proof.** By an integration by parts argument one and Bessel’s inequality one shows

\[
\sum_{n=-\infty}^{\infty} n^2 |\hat{g}(n)|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{d}{d\alpha} g(e^{i\alpha}) \right|^2 d\alpha < \infty.
\]
This inequality along with the Cauchy-Schwarz inequality, then shows
\[
\sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{g}(n)| = \sum_{n \in \mathbb{Z} \setminus \{0\}} |\hat{g}(n)| \cdot \frac{1}{|n|} 
\leq \left( \sum_{n=-\infty}^{\infty} n^2 |\hat{g}(n)|^2 \right)^{1/2} \cdot \left( \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} \right)^{1/2} < \infty.
\]
This shows the infinite sum in Eq. (69.16) is absolutely (hence uniformly convergent). Thus we pass to the limit as \( r \uparrow 1 \) in Eq. (69.14) in order to arrive at the equality in Eq. (69.16). \( \blacksquare \)