# 65

## Fluid Flows (6/1/2018)

Consider a two dimensional fluid flow which we describe by its velocity field,

$$V(x,y) = (p(x,y), q(x,y)) = p + iq \in \mathbb{R}^2.$$

We are only going to consider flows which are **incompressible**, i.e.

$$0 = \nabla \cdot V = p_x + q_y$$

and **irrotational**,

$$0 = \operatorname{curl}(V) = q_x - p_y.$$

*Example 65.1.* If $D$ is an open subset of $\mathbb{C}$ and $\Phi = \varphi + i\psi : D \to \mathbb{C}$ is an analytic function, then $V = (p,q) = \nabla\varphi = (\varphi_x, \varphi_y)$ is **incompressible and irrotational.** Indeed,

$$\begin{aligned} \nabla \cdot V &= \Delta\varphi = 0 \text{ and} \\ \operatorname{curl}(V) &= q_x - p_y = \varphi_{y,x} - \varphi_{x,y} = 0. \end{aligned}$$

**In fact;** the only thing we really need here is to know that $\varphi : D \to \mathbb{R}$ is a harmonic function. When $V = \nabla\varphi$ with $\Delta\varphi = 0$ we refer to $\varphi$ as a **potential function of** $V$.

**Lemma 65.2.** *If $V = (p,q) : D \to \mathbb{R}^2$ is a $C^1$ -function, then $V$ is incompressible and irrotational iff $f = p - iq$ is analytic on $D$ or equivalently iff $q + ip$ is analytic.*

**Proof.** This is a consequence of the Cauchy-Riemann equations and the identity,

$$\begin{aligned} f_y - if_x &= p_y - iq_y - i(p_x - iq_x) \\ &= p_y - q_x - i(p_x + q_y) = -[\operatorname{curl}(V) + i(\nabla \cdot V)]. \end{aligned}$$

■

**Theorem 65.3.** *If $D$ is a simply connected region and $V = (p,q) : D \to \mathbb{R}^2$ is a vector field, then $V$ is an incompressible irrotational flow iff there exists a complex potential, $\Phi = \varphi + i\psi : D \to \mathbb{C}$ such that $V = \nabla\varphi$.*

**Proof.** The easy and most important direction has already been discussed in Example 65.1 without the need for $D$ to be simply connected. For the converse direction, assume that $V$ is an incompressible irrotational flow so that $f = p - iq$ is an analytic function on $D$ by Lemma 65.2. By Corollary 64.9, there exists an analytic anti-derivative, $\Phi = \varphi + i\psi : D \to \mathbb{C}$, of $f$, i.e.

$$p - iq = f = \Phi' = \varphi_x + i\psi_x = \varphi_x - i\varphi_y$$

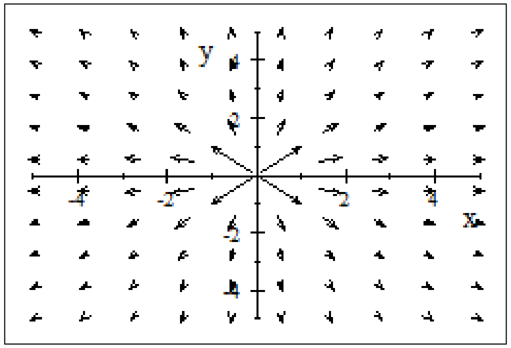
from which the result follows. ■

*Example 65.4.* If $\Phi(z) = Az$ where $A > 0$, then $\varphi(z) = Ax$ and hence $\nabla\varphi = (A, 0)$ is the uniform horizontal flow with speed $A$ and since $\nabla\varphi \cdot \nabla\psi = 0$ where $\psi(z) = Ay$, the flow lines are horizontal curves which is quite obvious.

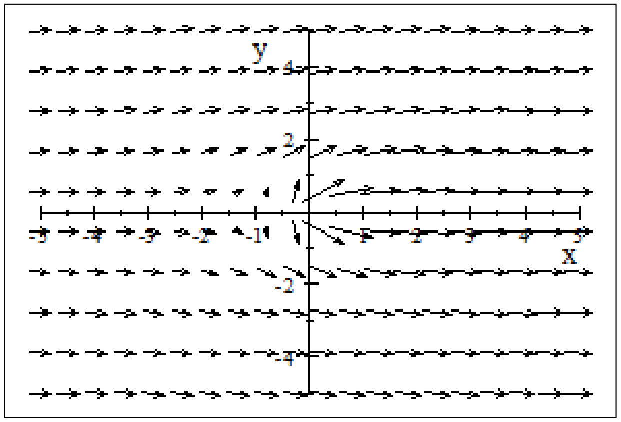
*Example 65.5.* Suppose that $\Phi(z) = \log(z)$ in which case $\varphi(z) = \ln|z|$ and $\psi(z) = \arg(z)$. In this case the vector field is given by

$$\nabla \ln\left(\sqrt{x^2+y^2}\right) = \frac{1}{x^2+y^2}(x,y).$$

If we add the two flows together we get




$$(A,0) + \frac{1}{x^2+y^2}(x,y).$$




See the mathematica notebook "fluid_flow_examples.nb" for pictures of these flows.

**Notation 65.6 (Flows and stream functions)** *Let $D$ be an open region in $\mathbb{C}$ and suppose that $\Phi = \varphi + i\psi : D \to \mathbb{C}$ is a complex potential and $V = \nabla\varphi$ is the associated irrotational and incompressible flow.*

1. *The trajectories of this flow are the solutions to the differential equation;*

$$\dot{\sigma}(t) = V(\sigma(t)) = \nabla\varphi(\sigma z(t)).$$

2. *The flow lines are the images of these trajectories which may be computed at the level curves of $\psi$. Indeed,*

$$\frac{d}{dt}\psi(\sigma(t)) = (\nabla\psi \cdot \nabla\varphi)(\sigma(t)) = 0$$

*because $\Phi$ is conformal or by the Cauchy Riemann equations,*

$$\nabla\psi \cdot \nabla\varphi = \psi_x \cdot \varphi_x - \psi_y\varphi_y = \pm(\varphi_y\varphi_x - \varphi_x\varphi_y) = 0.$$

3. *Since $\Phi' = \varphi_x + i\psi_x = \varphi_x - i\varphi_y$ it follows that*

$$V = p + iq = \overline{\Phi'}$$

*and in particular*

$$|V| = \left|\overline{\Phi'}\right| = |\Phi'|.$$

4. *The function $\varphi$ is called the* ***potential function of*** *$V$ while $\psi$ is the* ***stream function*** *of $V$.*

*Remark 65.7 (Flows in regions).* When looking for flows in a given region, $\Omega$, a key point is that the flows near a boundary of $\Omega$ ($\partial\Omega$) should be parallel to the boundary. In other words, the stream function, $\psi := \operatorname{Im}\Phi$, should vanish or at least be constant on the boundary of $\Omega$.

*Example 65.8.* We can transform the uniform flow to a wedge, say $Q_\theta := \left\{re^{i\alpha} : 0 \le r < \infty \text{ and } 0 \le \alpha \le \theta\right\}$ with $0 < \theta < \pi$ by considering the map,

$$g(z) := z^{\pi/\theta} = e^{\frac{\pi}{\theta}\operatorname{Log}(z)}.$$

In other words if $z = re^{i\alpha}$ in $Q_\theta$ then

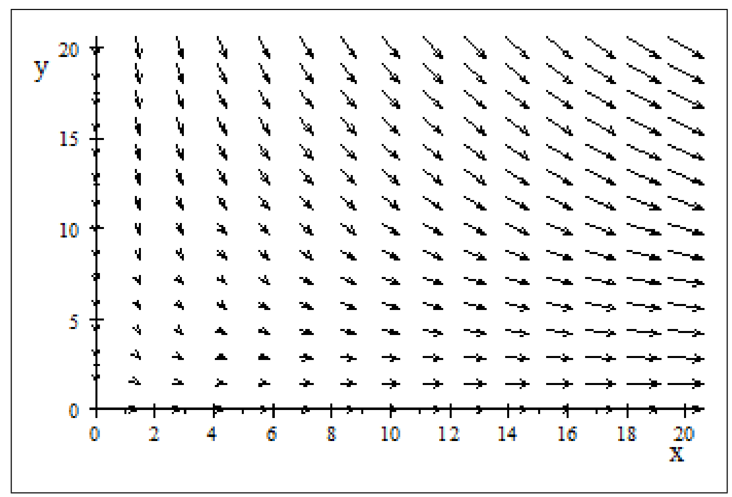
$$g(z) = r^{\frac{\pi}{\theta}} e^{i\alpha\frac{\pi}{\theta}}.$$

Then $g$ is conformal homeomorphism from $Q_\theta \to \mathbb{H}$ where $\mathbb{H}$ is the upper half plane with $g(\partial Q_\theta) = \partial\mathbb{H}$ which is a stream line for the horizontal flow. Thus we find $\Phi(z) = g(z)$ is the desired complex potential. For example when $\theta = \pi/2$ the we find

$$\Phi(z) = z^2 = x^2 - y^2 + i2xy = \varphi(x,y) + i\psi(x,y)$$

and the stream functions become,

$$\psi(x,y) = \operatorname{Im}\Phi(z) = 2xy.$$



# Fluids Continued (6/4/2018)

**Proposition 66.1.** *If $\Omega$ is an open set and $f:\Omega\to\mathbb{C}$ is analytic then $\tilde f:\bar\Omega\to\mathbb{C}$ defined by*

$$\tilde f(z)=\overline{f(\bar z)} \text{ for } z\in\bar\Omega$$

*is an analytic function and moreover,*

$$\tilde f'(z)=\overline{f'(\bar z)}.$$

**Proof.** Let $h\in\mathbb{C}$ be small and $z\in\bar\Omega$, then

$$\frac{\tilde f(z+h)-\tilde f(z)}{h}=\frac{\overline{f(\bar z+\bar h)}-\overline{f(\bar z)}}{h}$$
$$=\overline{\left(\frac{f(\bar z+\bar h)-f(\bar z)}{\bar h}\right)}\to\overline{f'(\bar z)} \text{ as } h\to 0$$

and this completes the proof. ■

**Theorem 66.2 (Milne-Thomson circle theorem).** *Let $f(z)$ be a complex potential with all of its singularities outside of $|z|=R$. Then*

$$\Phi(z):=f(z)+\overline{f\left(\frac{R^2}{\bar z}\right)} \tag{66.1}$$

*is a complex potential such that $\operatorname{Im}\Phi(z)=0$ when $|z|=R$ and moreover $\Phi(z)$ has the same singularities as $f(z)$ in $|z|>R$. [Note that for $|z|>>1$ that $\Phi(z)\sim f(z)+\overline{f(0)}$ and so the fluid flow associated to $\Phi$ and $f$ should be similar away from the inserted circular obstacle put into the flow.]*

**Proof.** We look for a function $g(z)$ which is analytic in a neighborhood of $|z|\ge\frac{1}{R}$ such that $\Phi(z):=f(z)+g(z)$ satisfies $\operatorname{Im}\Phi(z)=0$ when $|z|=R$. Now the condition that $\operatorname{Im}\Phi(z)=0$ when $|z|=R$ implies that

$$g(z)=\overline{f(z)} \text{ when } |z|=R. \tag{66.2}$$

Now we can not use $\overline{f(z)}$ as our function since it is not analytic!. However,

$$|z|=R\implies z\cdot\bar z=R^2\implies z=\frac{R^2}{\bar z}$$

and so we can write Eq. (66.2) as

$$g(z)=\overline{f\left(\frac{R^2}{\bar z}\right)} \text{ when } |z|=R.$$

By Proposition 66.1, $z\to\overline{f\left(\frac{R^2}{\bar z}\right)}$ is analytic and hence so is $\Phi(z)$ in Eq. (66.1). ■

*Example 66.3.* Suppose we take $f(z)=z$ and $R=1$ above then

$$J(z):=f(z)+\overline{f\left(\frac{1}{\bar z}\right)}=z+\frac{1}{z}$$
$$=x+iy+\frac{x-iy}{x^2+y^2}$$
$$=x\left(1+\frac{1}{x^2+y^2}\right)+iy\left(1-\frac{1}{x^2+y^2}\right).$$

The next few pictures indicate the streamlines associated to the flow coming form $\nabla\operatorname{Re}J$.

**Note relative to Problem 126.9:** these stream lines are consistent with a flow in the region

$$\mathbb{H}\setminus D=\{z\in\mathbb{C}:\operatorname{Im}z\ge 0 \text{ and } |z|\ge 1\}$$

since in this case

$$\psi(x,y)=\operatorname{Im}J(x+iy)=y\left(1-\frac{1}{x^2+y^2}\right)$$

vanishes on the boundary of $\mathbb{H}\setminus D$.

*Remark 66.4.* If $\Phi$ is a complex potential such that $\operatorname{Im}\Phi(z)=0$ if $|z|=1$ (i.e. is a complex potential on $\mathbb{C}\setminus D(0,1)$), then for $z_0\in\mathbb{C}$ and $R>0$,

$$\tilde\Phi(z):=\Phi\left(\frac{z-z_0}{R}\right)$$

a complex potential such that $\operatorname{Im}\tilde\Phi(z)=0$ if $|z-z_0|=R$, i.e. is a complex potential on $\mathbb{C}\setminus D(z_0,R)$. In particular if

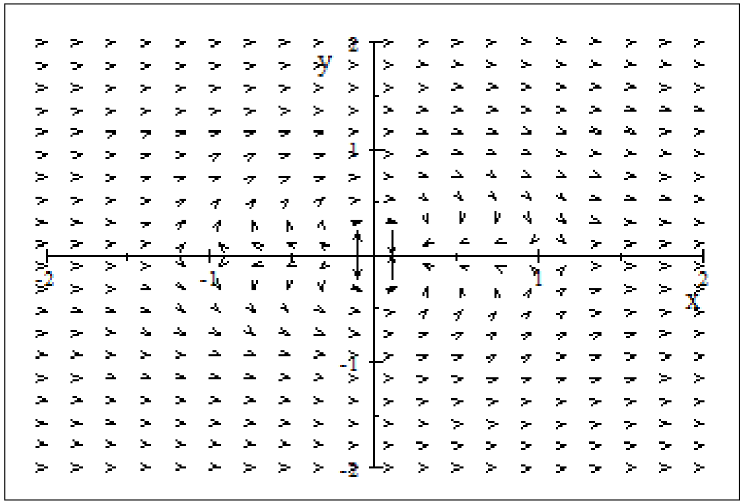



**Fig. 66.1.** The vector field, $\nabla \operatorname{Re} J$.

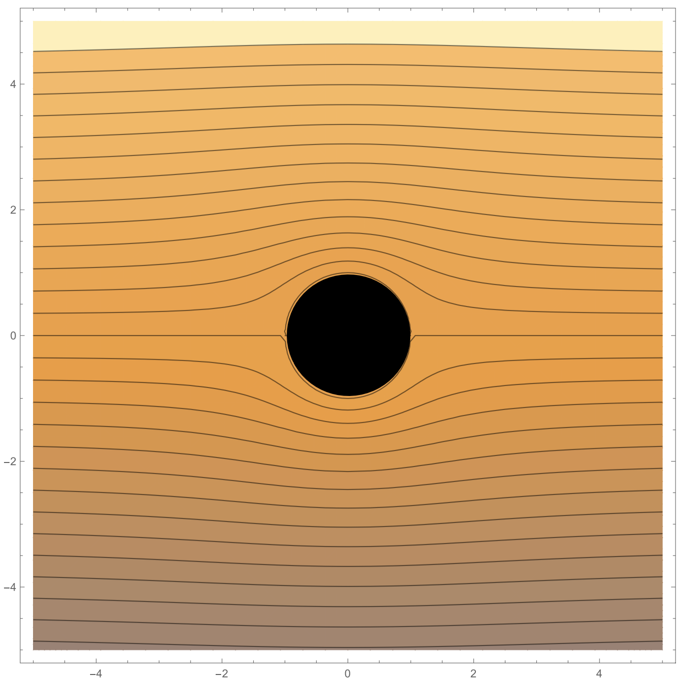



**Fig. 66.2.** A uniform flow around a circular region.

$$\Phi(z) = f(z) + \overline{f\left(\frac{1}{\bar{z}}\right)}$$

then

$$\tilde{\Phi}(z) := f\left(\frac{z - z_0}{R}\right) + \overline{f\left(\frac{R}{\bar{z} - \bar{z}_0}\right)}.$$

## 66.1 An introduction to airfoils

This section was not covered except by pictures in class.

**Definition 66.5 (Joukowski Mapping).** *The function $J(z) = z + 1/z$ is referred to as the Joukowski mapping.*

We now explore some interesting properties of this mapping related to airfoil design.

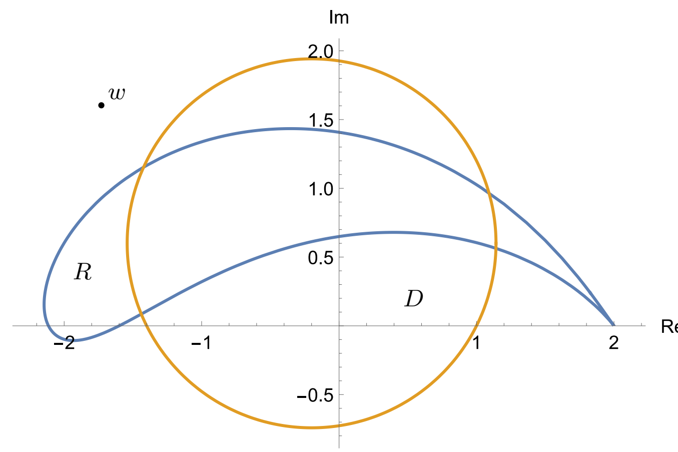



**Fig. 66.3.** A Joukowski airfoil (in blue) created by applying the Joukowski mapping to a circle $C$ (in orange) centered at $z_0 = a + ib$ which goes through 1.

**Proposition 66.6.** *Let $R$ be the cross section of the airfoil and $D = D(z_0, R)$ be the open disk bounded by the orange circle as indicated in Figure 66.3. Then $J$ maps $\mathbb{C} \setminus D$ onto $\mathbb{C} \setminus R$ and this map is one to one.*

**Proof.** Let us first observe that

$$J(z) = z + \frac{1}{z} = w \iff z^2 - zw + 1 = 0 \iff z = \frac{w \pm (w^2 - 4)}{2}$$

from which it follows that $J$ is a 2 to one map. This is true even when $w = \pm 2$ in which case $z = \pm 1$ where $J'(\pm 1) = 0$ since $J'(z) = 1 - z^{-1}$. The other key observations is that $J(z) := z + \frac{1}{z}$ has a pole of order 1 at zero. With this information in hand we may now use the argument principle to deduce the stated results. For this one checks that $\partial R$ is traversed once in the counterclockwise direction.

If $w \in R$, then

$$1 = N_{\partial R}(w) = \#\{z \in D : J(z) = w\} - 1$$

which implies $\#\{z \in D : J(z) = w\} = 2$ and hence both solutions are in $D$ and therefore $J(\mathbb{C} \setminus D) \subset \mathbb{C} \setminus R$. On the other hand, if $w \in \mathbb{C} \setminus R$ the argument principle implies

$$0 = N_{\partial R}(w) = \#\{z \in D : J(z) = w\} - 1$$

which implies that $J(z) = w$ has a unique solution in $D$ for each $w \in \mathbb{C} \setminus R$. Since $J(z)$ is 2 to 1 we infer that $J(z) = w \in \mathbb{C} \setminus R$ has exactly one solution in $\mathbb{C} \setminus D$ as well. Thus we have shown that $J$ maps $\mathbb{C} \setminus D$ onto $\mathbb{C} \setminus R$ in a one to one and onto fashion. ■

**Corollary 66.7.** *We will continuing the notation in Proposition 66.6. If $\Phi$ is a complex potential consistent with a flow in $\mathbb{C} \setminus D$ then $\Phi \circ J|_{\mathbb{C} \setminus D}^{-1}$ is a complex potential on $\mathbb{C} \setminus R$ and one may compute the streamlines of this potential using, with $c \in \operatorname{Im} \Phi(\mathbb{C} \setminus D)$,*

$$\left\{w : \operatorname{Im} \Phi \circ J|_{\mathbb{C} \setminus D}^{-1}(w) = c\right\} = \{J(z) : \operatorname{Im} \Phi(z) = c\}.$$

*That is we apply $J$ to the streamlines of $\Phi$ to get the streamlines associated with the Joukowski airfoil.*

We refer the interested reader to

$$https://www.grc.nasa.gov/WWW/K-12/airplane/map.html$$

and the links therein for more on this subject. Also see

$$http://demonstrations.wolfram.com/JoukowskiAirfoilFlowField/$$

for a simulator of the streamlines for various Joukowski airfoils.