Proposition 58.3. The map, 
\[
\begin{bmatrix} z \\ w \end{bmatrix} \rightarrow \frac{z}{w} \in \mathbb{C}_\infty
\]
is a bijection. The inverse map is given by 
\[
\mathbb{C}_\infty \ni z \rightarrow \left\{ \begin{array}{ll}
[z,1] & \text{if } z \neq 0 \\
[1,0] & \text{if } z = 0
\end{array} \right. \in \mathbb{CP}^2.
\]

Stereographic projection gives one useful way to represent and understand \(\mathbb{C}_\infty\). The next definition gives another particularly useful way to represent \(\mathbb{C}_\infty\) if \(A\) is a \(2 \times 2\) invertible matrix, then by matrix multiplication, 
\[
A \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} az + bw \\ cz + dw \end{bmatrix}
\]
and in particular, 
\[
A \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az + b \\ cz + d \end{bmatrix} = \begin{bmatrix} \frac{az + b}{cz + d} \\ 1 \end{bmatrix}
\]
and so we see that \(A\) induces the linear fractional transformation (LFT), 
\[
\psi_A : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \text{ defined by}
\]
\[
\psi_A (z) = \frac{az + b}{cz + d}.
\]

Remark 58.4. It is useful to note that \(\psi_{A\lambda} (z) = \psi_A (z)\) for all \(\lambda \in \mathbb{C} \setminus \{0\}\). Thus if we write \(A \sim B\) to mean that \(A = \lambda B\) for some \(\lambda \in \mathbb{C} \setminus \{0\}\), then \(\psi_A = \psi_B\).

Proposition 58.5. If \(A\) and \(B\) are two \(2 \times 2\) - invertible matrices, then \(\psi_A \circ \psi_B = \psi_{AB}\) and \(\psi_{A^{-1}} = \psi_A^{-1}\). Moreover, \(\psi_A\) is a composition of rotations, translations, dilations, and inversions, and hence takes circles in \(\mathbb{C}_\infty\) to circles in \(\mathbb{C}_\infty\) and these maps restricted from a circle to a circle are bijective. [As usual, a line in \(\mathbb{C}\) is considered to be a circle in \(\mathbb{C}_\infty\) which goes through \(\infty\).]

Proof. The assertion that \(\psi_A \circ \psi_B = \psi_{AB}\) follows from the construction described above. For the remaining assertion, recall that any \(2 \times 2\) invertible matrix may be written as a product of elementary matrices of the form
\[
\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
and the LFT’s associated to these three matrices are (respectively) given by
\[
\psi(z) = az, \quad \psi(z) = z + b, \quad \text{and } \psi(z) = \frac{1}{z}.
\]
Since we have already seen that each of these transformations takes a generalized circle to a generalized circle it follows that if \(C \subset \mathbb{C}_\infty\) is a generalized circle, then \(\psi(C) \subset C’\) for some other circle, \(C’ \subset \mathbb{C}_\infty\). Similarly there is a circle \(C'' \subset \mathbb{C}_\infty\) such that \(\psi^{-1}(C’’) \subset C''\) and hence,
\[
C = \psi^{-1}(\psi(C)) \subset \psi^{-1}(C’) \subset C’’.
\]
Since circles are determined by knowing three distinct points in the circle it follows that in fact \(C = C’’\) and so \(C = \psi^{-1}(C’)\). Applying \(\psi\) to this identity then shows \(\psi(C) = C’\).
Remark 58.6. If \( \psi(z) = \frac{az + b}{cz + d} \), then
\[
\psi^{-1}(z) = \frac{dz - b}{-cz + a}
\]
since
\[
A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \sim \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]
and so
\[
\psi^{-1}(z) = \psi_{A^{-1}}(z) = \frac{dz - b}{-cz + a}.
\]

Definition 58.7 (Cross Ratio). For \( a, b, c \) distinct point in \( \mathbb{C}_\infty \) we call
\[
(z, a, b, c) := S_{a,b,c}(z) = \frac{(z-a)(b-c)}{(z-c)(a-b)}
\]
the cross ratio with the following conventions:
\[
(z, \infty, b, c) := \lim_{a \to \infty} \frac{z-a}{z-c} = \frac{b-c}{b-a} = \frac{z - \infty)(b-c)}{(z-c)(b-a)}
\]
\[
(z, a, \infty, c) = \lim_{b \to \infty} \frac{z-a}{z-c} = \frac{b-c}{b-a} = \frac{z-a, \infty-c}{z-c, \infty-a}
\]
\[
(z, a, b, \infty) = \lim_{c \to \infty} \frac{z-a}{z-c} = \frac{b-c}{b-a} = \frac{z-a, \infty-b}{z-\infty, b-a}
\]
In other words, \( (z, a, b, c) \) is the unique LFT which takes \( a \to 0 \), \( b \to 1 \), and \( c \to \infty \).

Corollary 58.8. Suppose given \( (a, b, c) \in \mathbb{C}_\infty \) distinct and \( (a, \beta, \gamma) \in \mathbb{C}_\infty \) there exists \( \psi \in \{ \text{LFT's} \} \) such that \( \psi(a) = \alpha \), \( \psi(b) = \beta \), and \( \psi(c) = \gamma \). To construct this transformation, one should define \( \psi(z) = w \) where \( w \) is the unique solution to
\[
(w, \alpha, \beta, \gamma) = (z, a, b, c).
\]

Proof. Uniqueness. If \( T \) is another such fractional linear transformation, then \( T^{-1} \circ S \) is a fractional linear transformation fixing the the three distinct points, \( (a, b, c) \) and hence \( T^{-1} \circ S = \text{id} \), i.e., \( T = S \) by Proposition 59.3 below.

Existence. Define \( S_{a,b,c}(z) \) and \( S_{a,\beta,\gamma} \) and then
\[
\psi(z) := S_{a,\beta,\gamma}^{-1}(S_{a,b,c}(z))
\]
is the desired LFT. Furthermore, if we let \( w = \psi(z) \), then
\[
(w, \alpha, \beta, \gamma) = S_{a,\beta,\gamma}(w) = S_{a,b,c}(z) = (z, a, b, c).
\]

Example 58.9. Find the LFT, \( \psi \), such that \( \psi(-1) = -i \), \( \psi(0) = 1 \), and \( \psi(1) = i \).

Note well: \( \frac{1+i}{1-i} = \frac{1+i}{1-i} \).

To solve this problem we must solve the following identity for \( w \),
\[
(w, -i, 1, i) = (z, -1, 0, 1)
\]
where
\[
(z, -1, 0, 1) = \frac{-z + 1}{z - 1} = \frac{z + 1}{-z + 1} \approx A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}
\]
and
\[
(w, -i, 1, i) = \frac{w + i - i i}{w - i i} = \frac{-i w + i}{w - i} \approx B = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}.
\]
Thus we have to solve,
\[
-(w - i)(z + 1) = -i (z - 1) (w + i) \iff w \frac{-z - 1 + i (z - 1)}{w - 1 + i i} = -i (z - 1) - i (z + 1) = z - 1 - i (z + 1) = (1 - i) z - 1 - i
\]
and so
\[
w = \psi(z) = \frac{(1-i) z - (1+i)}{(i-1) z - (1+i)} = \frac{(\frac{-i z - 1}{i z - 1})}{\frac{iz + 1}{iz + 1}}.
\]
Alternatively using matrix calculations,
\[
B^{-1} A \approx \begin{bmatrix} -i & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 - i & -1 - i \\ 1 - i & 1 - i \end{bmatrix}
\]
and so again we find,
\[
\psi(z) = \frac{iz + 1}{-iz + 1}.
\]
Lets check this works:
\[
\psi(-1) = \frac{-i + 1}{i + 1} = -i, \ \psi(0) = 1, \ \text{and} \ \psi(1) = \frac{i + 1}{-i + 1} = i
\]
as desired.
(5/14/2018) Mapping Properties of LFT’s

Standing notation and known facts.

1. For all of this lecture, let \( \psi : \mathbb{C}_\infty \to \mathbb{C}_\infty \) be given by
   \[
   \psi(z) = \psi_A(z) = \frac{az + b}{cz + d}
   \]
   where
   \[
   A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2 \times 2} \quad \text{with} \quad \det A \neq 0.
   \]

2. Recall that \( \psi \) takes circles onto circles in \( \mathbb{C}_\infty \) and these maps are bijective on these (generalized) circles.

Remark 59.1 (On terminology). The transformations in Eq. (59.1) are referred to as linear fractional transformations, or bilinear transformations, or Möbius transformations. See Section 99 of the book for the reason \( \psi \) is called a bilinear transformation.

Here is the reason not covered in class: if \( w = \psi(z) \), then

\[
\begin{align*}
w &= \frac{az + b}{cz + d} \\
\iff w(cz + d) &= az + b \\
\iff cwz - az + dw - b &= 0 \\
\iff Azw + Bz + Cw + D &= 0 \quad \text{with} \quad AD - BC = -cb + a \cdot d = -\det A.
\end{align*}
\]

Proposition 59.3. If
\[
\psi(z) := \frac{az + b}{cz + d}
\]

is a fractional linear transformation which is not the identity, then \( S \) has either one or two fixed points in \( \mathbb{C}_\infty \), where \( z \in \mathbb{C}_\infty \), is a fixed point if \( S(z) = z \). [Hence if \( S \) has at least 3 fixed points then in fact \( S(z) = z \).]

Proof. Case (i) \( c \neq 0 \). In this case \( S(\infty) = a/c \in \mathbb{C} \) and hence \( \infty \) is not a fixed point of \( S \). For \( z \in \mathbb{C} \) we have \( S(z) = z \) if

\[
\begin{align*}
ax + b &= z(cz + d) = cz^2 + dz \\
\iff cz^2 + (d - a)z - b &= 0.\tag{59.3}
\end{align*}
\]

This quadratic polynomial can have at most two solutions and hence \( S \) has at most two fixed points in this case.\(^1\)

\(^1\) Note that if \( c = 1 = -b \) and \( d = 2 \) and \( a = 0 \), then

\[
S(z) = \frac{-1}{z + 2} = z \iff -1 = z^2 + 2z \iff z^2 + 2z + 1 = 0 \iff z = 1,
\]

which shows that it is possible to have only one fixed point.
Case (ii) \( c = 0 \). In this case \( a \cdot d \neq 0 \) and \( S(\infty) = \infty \) so that \( \infty \) is a fixed point. Moreover, \( z \in \mathbb{C} \) is a fixed point iff
\[
ag + bq = z \cdot d \iff (d - a) z = b
\]
and hence \( z = \frac{b}{d - a} \) is the only other fixed point when \( d \neq a \). If \( d = a \) and \( b \neq 0 \) then the above equation has no solutions and hence \( \infty \) is the only fixed point.

If \( d = a \) and \( b = 0 \), then in fact \( S(z) = z \) and every point is a fixed point.

59.1 Mapping properties of certain LFT’s

Theorem 59.4 (LFT taking \( \mathbb{R}_\infty \) to \( \mathbb{R}_\infty \)). A LFT, \( \psi \), takes \( \mathbb{R}_\infty \) to \( \mathbb{R}_\infty \) iff there exists \( a, b, c, d \in \mathbb{R} \) so that
\[
\psi(z) = \psi_A(z) = \frac{az + b}{cz + d}.
\]
Moreover such a \( \psi \) will take the upper half plane to the upper half plane iff
\[
det A = a \cdot d - b \cdot c > 0.
\]

Proof. Clearly if \( a, b, c, d \in \mathbb{R} \) then \( \psi_A(x) \in \mathbb{R}_\infty \) for all \( x \in \mathbb{R}_\infty \), so now assume that \( \psi \) takes \( \mathbb{R}_\infty \) to \( \mathbb{R}_\infty \). Let \( \alpha = \psi^{-1}(0) \in \mathbb{R}_\infty \), \( \beta = \psi^{-1}(1) \in \mathbb{R}_\infty \), and \( \gamma = \psi^{-1}(\infty) \), then \( \psi(z) = (z, \alpha, \beta, \gamma) \) is of the form in Eq. (59.4) with all coefficients being real.

If we further wish to have \( \psi \) take the upper half plane to itself we must require \( \text{Im}(\psi(i)) > 0 \). However,
\[
\text{Im}(\psi(i)) = \text{Im}(\frac{(ai + b) - ci + d}{c^2 + d^2}) = \frac{a \cdot d - b \cdot c}{c^2 + d^2} = \frac{\det A}{c^2 + d^2}.
\]

\( S \) below is the unit circle centered at 0 in the complex plane.

Theorem 59.5 (FLT taking \( S \) to \( \mathbb{R}_\infty \)). The general form of a LFT (\( \psi \)) which takes \( S \) to \( \mathbb{R}_\infty \) is
\[
\psi(z) = \frac{\xi z + \bar{\xi}}{wz + \bar{w}}.
\]
where \( \xi, w \in \mathbb{C} \) such that \( \text{Im}(\xi \bar{w}) \neq 0 \). If we write \( \xi = r e^{i\theta} \) and \( w = re^{i\alpha} \) and \( k = r / \rho > 0 \), then
\[
\psi(z) = k \frac{ze^{i\theta} + e^{-i\theta}}{ze^{i\alpha} + e^{-i\alpha}}.
\]

Proof. The LFT,
\[
S(z) := (z, 1, i, -1) = \frac{z - 1 + i - i}{z + 1 + i - i} = \frac{i z + i}{z + 1},
\]
takes \( S \) to \( \mathbb{R}_\infty \) and so the general such LFT is of the form \( \varphi \circ S \) where
\[
\varphi(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{R}.
\]
The matrix associated with this LFT is
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
-i i \\
1 1
\end{pmatrix} = \begin{pmatrix}
-b - ia & b + ia \\
d - ic & d + ic
\end{pmatrix} = \begin{pmatrix}
\xi & \bar{\xi} \\
\bar{w} \bar{z} & w \bar{z}
\end{pmatrix}
\]
and so the general form of \( \psi \) is as in Eq. (59.5) or equivalently the form in Eq. (59.6). Conversely if \( \psi \) is given as in Eq. (59.5) and \( |z| = 1 \), then
\[
\psi(z) = \frac{\xi z + \bar{\xi}}{wz + \bar{w}} = \frac{\xi z + \xi}{wz + \bar{w}} = \frac{\xi + \xi z}{w + wz} = \psi(z)
\]
which shows \( \psi(z) \in \mathbb{R}_\infty \) as claimed.

Corollary 59.6 (FLT taking \( \mathbb{R}_\infty \) to \( S \)). The general LFT (\( \psi \)) which takes \( \mathbb{R}_\infty \) to \( S \) may be written as
\[
\psi(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \bar{z}_0}
\]
for some \( \theta \in \mathbb{R} \) and \( z_0 \in \mathbb{C} \) with \( \text{Im}(z_0) \neq 0 \). Moreover, if we want the upper half plane to go to the interior of \( S \), then we must require \( \text{Im}(z_0) > 0 \), i.e. that \( z_0 \) be in the upper half plane.

[Note: it is simple to observe if \( z = x \in \mathbb{R} \) and \( \psi \) is given as in Eq. (59.7), then \( |\psi(x)| = 1 \).]

Proof. The general form of the LFT we are looking for is the inverse of a \( \psi \) given in Eq. (59.5), i.e. of the form,
\[
z \rightarrow \frac{\bar{w} z - \bar{\xi}}{wz + \xi} = -\frac{\bar{w}}{w} \cdot \frac{z - \xi / \bar{w}}{z - \xi / w} = e^{i\theta} \cdot \frac{z - z_0}{z - \bar{z}_0}.
\]
The last assertion now easily follows from the fact that \( \psi(z_0) = 0 \).
Theorem 59.7 (FLT taking $S$ to $S$). The general LFT ($\psi$) which takes $S$ to $S$ may be written as

$$\psi(z) = e^{i\alpha} \cdot \frac{z - \zeta}{1 - \zeta \bar{z}}$$

(59.8)

for some $\alpha \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ with $|\zeta|^2 \neq 1$. If $\psi$ is to take the interior of $S$ to itself we must further require that $|\zeta| < 1$. [Note again that if $\psi$ is as above and $|z| = 1$, then]

$$|\psi(z)| = \left|\frac{z - \zeta}{1 - \zeta \bar{z}}\right| = \frac{|z - \zeta|}{|\bar{z} - \zeta|} = 1.$$

Proof. A particular LFT taking $S \to \mathbb{R}_\infty$ is given by

$$z \to -i \frac{z - 1}{z + 1} = \frac{-iz + i}{z + 1} \sim \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

While the general LFT taking $\mathbb{R}_\infty \to S$ is a composition of a rotation, $e^{i\theta}$, and an LFT of the form,

$$z \to z \frac{z - z_0}{z - \bar{z}_0} \sim \begin{bmatrix} 1 - z_0 & \bar{z}_0 \\ 1 - \bar{z}_0 & 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 - z_0 & -i \\ 1 - \bar{z}_0 & 1 \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -z_0 - i & -z_0 \\ -\bar{z}_0 - i & -\bar{z}_0 \end{bmatrix} \sim \begin{bmatrix} z_0 + i & z_0 - i \\ \bar{z}_0 + i & \bar{z}_0 - i \end{bmatrix}$$

the general LFT preserving $S$ is of the form,

$$\psi(z) = e^{i\theta} \frac{(z_0 + i)z + z_0 - i}{(\bar{z}_0 + i)z + \bar{z}_0 - i}$$

$$= e^{i\theta} \frac{z_0 + i}{\bar{z}_0 - i} \frac{z + \frac{z_0 - i}{z_0 + i}}{z + \frac{\bar{z}_0 + i}{\bar{z}_0 - i}}$$

$$= e^{i\alpha} \cdot \frac{z - \zeta}{-\zeta \bar{z} + 1} \text{ where } \zeta := -\frac{z_0 - i}{\bar{z}_0 + i}.$$