(5/11/2018) Linear Fractional Transformations

Definition 58.1 (Riemann Sphere). *The **Riemann sphere** ($\mathbb{C}_\infty$) is $\mathbb{C}$ with an added point denoted by "∞." By convention we let*

$$\frac{1}{0} = \infty \in \mathbb{C}_\infty \text{ and } \frac{1}{\infty} = 0 \in \mathbb{C} \subset \mathbb{C}_\infty.$$

Stereographic projection gives one useful way to represent and understand $\mathbb{C}_\infty$. The next definition gives another particularly useful way to represent $\mathbb{C}_\infty$.

Definition 58.2 (Complex Projective Space). *Complex projective spaces $\left(\mathbb{CP}^2\right)$ is the set of lines in $\mathbb{C}^2$. More explicitly, for $(z,w) \in \mathbb{C}^2 \setminus \{0\}$, then $[z,w] := \mathbb{C} \cdot (z,w)$ is the line containing (z,w) and hence an element of $\mathbb{CP}^2$.*

The easy proof of the following proposition is left to the reader.

Proposition 58.3. *The map,*

$$\mathbb{CP}^2 \ni [z,w] \to \frac{z}{w} \in \mathbb{C}_\infty$$

is a bijection. The inverse map is given by

$$\mathbb{C}_\infty \ni z \to \begin{cases} [z,1] \text{ if } z \neq 0 \\ [1,0] \text{ if } z = 0 \end{cases} \in \mathbb{CP}^2.$$

[Formally, $[\infty,1] = \frac{1}{\infty}[\infty,1] = [1,0]$.]

If A is a 2×2 invertible matrix, then by matrix multiplication, A takes lines to lines, i.e. defines a map $A : \mathbb{CP}^2 \to \mathbb{CP}^2$. More explicitly, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } \det A \neq 0,$$

then

$$A\begin{bmatrix} z \\ w \end{bmatrix} = \left[A\begin{pmatrix} z \\ w \end{pmatrix}\right] = \begin{bmatrix} az+bw \\ cz+dw \end{bmatrix}$$

and in particular,

$$A\begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ cz+d \end{bmatrix} = \begin{bmatrix} \frac{az+b}{cz+d} \\ 1 \end{bmatrix}$$

and so we see that A induces the **linear fractional transformation (LFT),** $\psi_A : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by

$$\psi_A(z) = \frac{az+b}{cz+d}.$$

Remark 58.4. It is useful to note that $\psi_{\lambda A}(z) = \psi_A(z)$ for all $\lambda \in \mathbb{C}\setminus\{0\}$. Thus if we write $A \sim B$ to mean that $A = \lambda B$ for some $\lambda \in \mathbb{C} \setminus \{0\}$, then $\psi_A = \psi_B$.

Proposition 58.5. *If A and B are two 2×2 - invertible matrices, then $\psi_A \circ \psi_B = \psi_{AB}$ and $\psi_{A^{-1}} = \psi_A^{-1}$. Moreover, ψ_A is a composition of rotations, translations, dilations, and inversions, and hence takes circles in $\mathbb{C}_\infty$ to circles in $\mathbb{C}_\infty$ and these maps restricted from a circle to a circle are bijective. [As usual, a line in $\mathbb{C}$ is considered to be a circle in $\mathbb{C}_\infty$ which goes through ∞.]*

Proof. The assertion that $\psi_A \circ \psi_B = \psi_{AB}$ follows from the construction described above. For the remaining assertion, recall that any 2×2 invertible matrix may be written as a product of elementary matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the LFT's associated to these three matrices are (respectively) given by

$$\psi(z) = az, \quad \psi(z) = z+b, \text{ and } \psi(z) = \frac{1}{z}.$$

Since we have already seen that each of these transformations takes a generalized circle to a generalized circle it follows that if $C \subset \mathbb{C}_\infty$ is a generalized circle, then $\psi(C) \subset C'$ for some other circle, $C' \subset \mathbb{C}_\infty$. Similarly there is a circle $C'' \subset \mathbb{C}_\infty$ such that $\psi^{-1}(C') \subset C''$ and hence,

$$C = \psi^{-1}(\psi(C)) \subset \psi^{-1}(C') \subset C''.$$

Since circles are determined by knowing three distinct points in the circle it follows that in fact $C = C''$ and so $C = \psi^{-1}(C')$. Applying ψ to this identity then shows $\psi(C) = C'$. ∎

Remark 58.6. If $\psi(z) = \frac{az+b}{cz+d}$, then

$$\psi^{-1}(z) = \frac{dz-b}{-cz+a}$$

since

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \sim \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and so

$$\psi^{-1}(z) = \psi_{A^{-1}}(z) = \frac{dz-b}{-cz+a}.$$

Definition 58.7 (Cross Ratio). *For a, b, c distinct point in $\mathbb{C}_\infty$ we call*

$$(z,a,b,c) := S_{a,b,c}(z) = \frac{(z-a)(b-c)}{(z-c)(a-b)}$$

*the **cross ratio** with the following conventions;*

$$(z,\infty,b,c) := \lim_{a\to\infty} \frac{z-a}{z-c}\cdot\frac{b-c}{b-a} = \frac{b-c}{z-c} = \text{“}\frac{(z-\infty)(b-c)}{(z-c)(b-\infty)}\text{”}$$

$$(z,a,\infty,c) = \lim_{b\to\infty} \frac{z-a}{z-c}\cdot\frac{b-c}{b-a} = \frac{z-a}{z-c} = \text{“}\frac{z-a}{z-c}\cdot\frac{\infty-c}{\infty-a}\text{”}$$

$$(z,a,b,\infty) = \lim_{c\to\infty} \frac{z-a}{z-c}\cdot\frac{b-c}{b-a} = \frac{z-a}{b-a} = \text{“}\frac{z-a}{z-\infty}\cdot\frac{b-\infty}{b-a}\text{”}.$$

In other words, (z,a,b,c) is the unique LFT which takes $a\to 0$, $b\to 1$, and $c\to\infty$.

Corollary 58.8. *Suppose given $(a,b,c)\in\mathbb{C}_\infty$ distinct and $(\alpha,\beta,\gamma)\in\mathbb{C}_\infty$ there exists $\psi\in\{LFT's\}$ such that $\psi(a)=\alpha$, $\psi(b)=\beta$, and $\psi(c)=\gamma$. To construct this transformation, one should define $\psi(z)=w$ where w is the unique solution to*

$$(w,\alpha,\beta,\gamma) = (z,a,b,c).$$

Proof. Uniqueness. If T is another such fractional linear transformation, then $T^{-1}\circ S$ is a fractional linear transformation fixing the the three distinct points, (a,b,c) and hence $T^{-1}\circ S = id$, i.e. $T=S$ by Proposition 59.3 below.

Existence. Define $S_{a,b,c}(z)$ and $S_{\alpha,\beta,\gamma}$ and then

$$\psi(z) := S^{-1}_{\alpha,\beta,\gamma}(S_{a,b,c}(z))$$

is the desired LFT. Furthermore, if we let $w=\psi(z)$, then

$$(w,\alpha,\beta,\gamma) = S_{\alpha,\beta,\gamma}(w) = S_{a,b,c}(z) = (z,a,b,c).$$

■

Example 58.9. Find the LFT, ψ, such that $\psi(-1)=-i$, $\psi(0)=1$, and $\psi(1)=i$.

$$\text{Note well: } \frac{1+i}{1-i} = \frac{1+i}{1-i}\frac{i}{i} = i.$$

To solve this problem we must solve the following identity for w,

$$(w,-i,1,i) = (z,-1,0,1)$$

where

$$(z,-1,0,1) = -\frac{z+1}{z-1} = \frac{z+1}{-z+1} \leftrightsquigarrow A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$(w,-i,1,i) = \frac{w+i}{w-i}\frac{1-i}{1+i}\frac{i}{i} = -i\frac{w+i}{w-i} \leftrightsquigarrow B = \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}.$$

Thus we have to solve,

$$\begin{aligned} -(w-i)(z+1) &= -i(z-1)(w+i) \iff \\ w[-z-1+i(z-1)] &= -i(z-1)i - i(z+1) \\ &= z-1-i(z+1) = (1-i)z-1-i \end{aligned}$$

and so

$$w = \psi(z) = \frac{(1-i)z-(1+i)}{(i-1)z-(1+i)} = \frac{\left(\frac{1-i}{1+i}\right)z-1}{-\left(\frac{1-i}{1+i}\right)z-1}$$
$$= \frac{-iz-1}{iz-1} = \frac{iz+1}{-iz+1}.$$

Alternatively using matrix calculations,

$$\begin{aligned} B^{-1}A &\sim \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-i & -1-i \\ -1+i & -1-i \end{bmatrix} \\ &\sim \begin{bmatrix} -\frac{1-i}{1+i} & 1 \\ -\frac{-1+i}{1+i} & 1 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \end{aligned}$$

and so again we find,

$$\psi(z) = \frac{iz+1}{-iz+1}.$$

Lets check this works:

$$\psi(-1) = \frac{-i+1}{i+1} = -i,\ \psi(0)=1,\ \text{and } \psi(1) = \frac{i+1}{-i+1} = i$$

as desired.

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(5/14/2018) Mapping Properties of LFT's

Standing notation and known facts.

1. For all of this lecture, let $\psi : C_\infty \to C_\infty$ be given by
$$\psi(z) = \psi_A(z) = \frac{az+b}{cz+d} \tag{59.1}$$
where
$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{C}^{2\times 2} \text{ with } \det A \neq 0.$$
2. Recall that ψ takes circles onto circles in $\mathbb{C}_\infty$ and these maps are bijective on these (generalized) circles.

Remark 59.1 (On terminolgy). The transformations in Eq. (59.1) are referred to as **linear fractional transformations, or bilinear transformations, or Möbius transformations.** See Section 99 of the book for the reason ψ is called a bilinear transformation.

Here is the reason not covered in class: if $w = \psi(z)$, then

$$w = \frac{az+b}{cz+d} \iff w(cz+d) = az+b \iff cwz - az + dw - b = 0$$
$$\iff Azw + Bz + Cw + D = 0 \text{ with } AD - BC = -cb + a\cdot d = -\det A.$$

Example 59.2. Using Corollary 58.8, find the unique LFT so that $\psi(1) = i$, $\psi(0) = \infty$, and $\psi(-1) = 1$. To this end we must again solve for w the equation,
$$(w, i, \infty, 1) = (z, 1, 0, -1)$$
where
$$(z, 1, 0, -1) = -\frac{z-1}{z+1} \leftrightsquigarrow \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = A$$
and
$$(w, i, \infty, 1) = \frac{w-i}{w-1}\frac{\infty - 1}{\infty - i} = \frac{w-i}{w-1} \leftrightsquigarrow \begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix} = B.$$
Solving for w gives;
$$\psi(z) = \frac{(1+i)z + (i-1)}{2z}. \tag{59.2}$$

Here is the algebra. We need to solve for w,
$$-\frac{z-1}{z+1} = \frac{w-i}{w-1} \iff (z+1)(w-i) = (z-1)(1-w)$$
$$\iff w[z+1+z-1] = z - 1 + i(z+1) = (1+i)z + (i-1)$$
which easily gives Eq. (59.2).

Alternatively by matrix methods:
$$B^{-1}A = \begin{bmatrix} 1 & -i \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1+i & -1+i \\ 2 & 0 \end{bmatrix}.$$

The next proposition shows that ψ was used to guarantee that the constructions described in Corollary 58.8 is unique.

Proposition 59.3. *If*
$$S(z) := \frac{az+b}{cz+d}$$
is a fractional linear transformation which is ***not*** *the identity, then S has either one or two fixed points in $\mathbb{C}_\infty$ where $z \in \mathbb{C}_\infty$ is a fixed point iff $S(z) = z$. [Hence if S has at least 3 fixed points then in fact $S(z) = z$.]*

Proof. Case (i) $c \neq 0$. In this case $S(\infty) = a/c \in \mathbb{C}$ and hence ∞ is not be a fixed point of S. For $z \in \mathbb{C}$ we have $S(z) = z$ iff
$$az + b = z(cz+d) = cz^2 + dz \iff cz^2 + (d-a)z - b = 0. \tag{59.3}$$
This quadratic polynomial can have at most two solutions and hence S has at most two fixed points in this case.[1]

[1] Note that if $c = 1 = -b$ and $d = 2$ and $a = 0$, then
$$S(z) = \frac{-1}{z+2} = z \iff -1 = z^2 + 2z \iff z^2 + 2z + 1 = 0 \iff z = 1,$$
which shows that it is possible to have only one fixed point.

Case (ii) $c = 0$. In this case $a \cdot d \neq 0$ and $S(\infty) = \infty$ so that ∞ is a fixed point. Moreover, $z \in \mathbb{C}$ is a fixed point iff

$$az + b = z \cdot d \iff (d - a) z = b$$

and hence $z = \frac{b}{d-a}$ is the only other fixed point when $d \neq a$. If $d = a$ and $b \neq 0$ then the above equation has no solutions and hence ∞ is the only fixed point. If $d = a$ and $b = 0$, then in fact $S(z) = z$ and every point is a fixed point. ■

59.1 Mapping properties of certain LFT's

Theorem 59.4 (LFT taking $\mathbb{R}_\infty$ to $\mathbb{R}_\infty$). *A LFT, ψ, takes $\mathbb{R}_\infty$ to $\mathbb{R}_\infty$ iff there exists $a, b, c, d \in \mathbb{R}$ so that*

$$\psi(z) = \psi_A(z) = \frac{az + b}{cz + d}. \tag{59.4}$$

Moreover such a ψ will take the upper half plane to the upper half plane iff

$$\det A = a \cdot d - b \cdot c > 0.$$

Proof. Clearly if $a, b, c, d \in \mathbb{R}$ then $\psi_A(x) \in \mathbb{R}_\infty$ for all $x \in \mathbb{R}_\infty$, so now assume that ψ takes $\mathbb{R}_\infty$ to $\mathbb{R}_\infty$. Let $\alpha = \psi^{-1}(0) \in \mathbb{R}_\infty$, $\beta = \psi^{-1}(1) \in \mathbb{R}_\infty$, and $\gamma = \psi^{-1}(\infty)$, then $\psi(z) = (z, \alpha, \beta, \gamma)$ is of the form in Eq. (59.4) with all coefficients being real.

If we further wish to have ψ take the upper half plane to itself we must require $\operatorname{Im} \psi(i) > 0$. However,

$$\begin{aligned}\operatorname{Im} \psi(i) &= \operatorname{Im} \frac{(ai + b)(-ci + d)}{c^2 + d^2} \\ &= \frac{a \cdot d - b \cdot c}{c^2 + d^2} = \frac{\det A}{c^2 + d^2}.\end{aligned}$$

S below is the unit circle centered at 0 in the complex plane. ■

Theorem 59.5 (FLT taking S to $\mathbb{R}_\infty$). *The general form of a LFT (ψ) which takes S to $\mathbb{R}_\infty$ is*

$$\psi(z) = \frac{\xi z + \bar{\xi}}{wz + \bar{w}} \tag{59.5}$$

where $\xi, w \in \mathbb{C}$ such that $\operatorname{Im}(\xi \bar{w}) \neq 0$. If we write $\xi = re^{i\theta}$ and $w = \rho e^{i\alpha}$ and $k = r/\rho > 0$, then

$$\psi(z) = k \frac{ze^{i\theta} + e^{-i\theta}}{ze^{i\alpha} + e^{-i\alpha}}. \tag{59.6}$$

Proof. The LFT,

$$\begin{aligned}S(z) := (z, 1, i, -1) &= \frac{z - 1}{z + 1} \frac{i + 1}{i - 1} \frac{-i}{-i} \\ &= -i \frac{z - 1}{z + 1} = \frac{-iz + i}{z + 1},\end{aligned}$$

takes S to $\mathbb{R}_\infty$ and so the general such LFT is of the form $\varphi \circ S$ where

$$\varphi(z) = \frac{az + b}{cz + d} \text{ with } a, b, c, d \in \mathbb{R}.$$

The matrix associated with this LFT is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b - ia & b + ia \\ d - ic & d + ic \end{pmatrix} = \begin{pmatrix} \xi & \bar{\xi} \\ w & \bar{w} \end{pmatrix}$$

and so the general form of ψ is as in Eq. (59.5) or equivalently the form in Eq. (59.6). Conversely if ψ is given as in Eq. (59.5) and $|z| = 1$, then

$$\overline{\psi(z)} = \overline{\frac{\xi z + \bar{\xi}}{wz + \bar{w}}} = \frac{\bar{\xi} \bar{z} + \xi}{\bar{w} \bar{z} + w} = \frac{\bar{\xi} \bar{z} + \xi}{\bar{w} \bar{z} + w} \frac{z}{z} = \frac{\bar{\xi} + \xi z}{\bar{w} + wz} = \psi(z)$$

which shows $\psi(z) \in \mathbb{R}_\infty$ as claimed. ■

Corollary 59.6 (FLT taking $\mathbb{R}_\infty$ to S). *The general LFT (ψ) which takes $\mathbb{R}_\infty$ to S may be written as*

$$\psi(z) = e^{i\theta} \cdot \frac{z - z_0}{z - \bar{z}_0} \tag{59.7}$$

for some $\theta \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ with $\operatorname{Im} z_0 \neq 0$. Moreover, if we want the upper half plane to go to the interior of S, then we must require $\operatorname{Im} z_0 > 0$, i.e. that z_0 be in the upper half plane.

*[**Note:** it is simple to observe if $z = x \in \mathbb{R}$ and ψ is given as in Eq. (59.7), then $|\psi(x)| = 1$.]*

Proof. The general form of the LFT we are looking for is the inverse of a ψ given in Eq. (59.5), i.e. of the form,

$$z \to \frac{\bar{w} z - \bar{\xi}}{-wz + \xi} = -\frac{\bar{w}}{w} \cdot \frac{z - \bar{\xi}/\bar{w}}{z - \xi/w} = e^{i\theta} \cdot \frac{z - z_0}{z - \bar{z}_0}.$$

The last assertion now easily follows from the fact that $\psi(z_0) = 0$. ■

Theorem 59.7 (FLT taking S to S). *The general LFT (ψ) which takes S to S may be written as*

$$\psi(z) = e^{i\alpha} \cdot \frac{z-\zeta}{1-\bar{\zeta}z} \tag{59.8}$$

for some $\alpha \in \mathbb{R}$ and $\zeta \in \mathbb{C}$ with $|\zeta|^2 \neq 1$. If ψ is to take the interior of S to itself we must further require that $|\zeta| < 1$. [Note again that if ψ is as above and $|z| = 1$, then

$$|\psi(z)| = \left|\frac{z-\zeta}{1-\bar{\zeta}z}\right| \frac{1}{|\bar{z}|} = \left|\frac{z-\zeta}{\bar{z}-\bar{\zeta}}\right| = 1.$$

Proof. A particular LFT taking $S \to \mathbb{R}_\infty$ is given by

$$z \to -i\frac{z-1}{z+1} = \frac{-iz+i}{z+1} \sim \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$$

While the general LFT taking $\mathbb{R}_\infty \to S$ is a composition of a rotation, $e^{i\theta}$, and an LFT of the form,

$$z \to \frac{z-z_0}{z-\bar{z}_0} \sim \begin{bmatrix} 1 & -z_0 \\ 1 & -\bar{z}_0 \end{bmatrix}.$$

Since

$$\begin{aligned} \begin{bmatrix} 1 & -z_0 \\ 1 & -\bar{z}_0 \end{bmatrix} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} &= \begin{bmatrix} -z_0 - i & i - z_0 \\ -\bar{z}_0 - i & i - \bar{z}_0 \end{bmatrix} \\ &\sim \begin{bmatrix} z_0 + i & z_0 - i \\ \bar{z}_0 + i & \bar{z}_0 - i \end{bmatrix} \end{aligned}$$

the general LFT preserving S is of the form,

$$\begin{aligned} \psi(z) &= e^{i\theta} \frac{(z_0+i)\,z + z_0 - i}{(\bar{z}_0+i)\,z + \bar{z}_0 - i} \\ &= e^{i\theta} \frac{z_0+i}{\bar{z}_0 - i} \frac{z + \frac{z_0-i}{z_0+i}}{\left(\frac{\bar{z}_0+i}{\bar{z}_0-i}\right) z + 1} \\ &= e^{i\alpha} \cdot \frac{z-\zeta}{-\bar{\zeta}z+1} \text{ where } \zeta := -\frac{z_0-i}{z_0+i}. \end{aligned}$$

∎