49

(4/20/2018)

49.1 Winding Numbers

If $\sigma : [a,b] \to \mathbb{C}$ is a curve in $\mathbb{C}$ and w is not in the image of σ, then we may find differentiable functions, $r(t) > 0$ and $\theta(t) \in \mathbb{R}$ such that

$$\sigma(t) = w + r(t)\, e^{i\theta(t)}. \tag{49.1}$$

The function $\theta(t)$ is unique up to an additive constant, $2\pi k$, for some $k \in \mathbb{Z}$ and therefore

$$\Delta \arg(\sigma) := \Delta\theta = \theta(b) - \theta(a)$$

is well defined independent of the choice of θ. Moreover, if σ is a loop so that $\sigma(b) = \sigma(a)$, then $e^{i\theta(b)} = e^{i\theta(a)}$ which implies that $\theta(b) - \theta(a) = 2\pi \cdot n$ for some $n \in \mathbb{Z}$.

Definition 49.1. *Given a continuous loop $\sigma : [a,b] \to \mathbb{C}$ and w not in the image of σ, the* ***winding number*** *of σ around w is defined by*

$$N_\sigma(w) := \frac{1}{2\pi}\Delta \arg(\sigma) := \frac{1}{2\pi}[\theta(b) - \theta(a)] \in \mathbb{Z}$$

where $\theta(t)$ is any continuous choice of angle so that Eq. (49.1) holds with $r(t) = |\sigma(t) - w|$.

Remark 49.2. It follows from the definition that

$$N_\sigma(w) = N_{\sigma - w}(0).$$

Indeed it σ is written as in Eq. (49.1), then

$$\sigma(t) - w = 0 + r(t)\, e^{i\theta(t)} \implies N_{\sigma-w}(0) = \frac{1}{2\pi}\Delta\theta = N_\sigma(w).$$

Example 49.3. Suppose that $\sigma(t) = e^{it}$ with $0 \le t \le n2\pi$, then $N_\sigma(0) = n$ while if $\sigma(t) = e^{-it}$ with $0 \le t \le n2\pi$, then $N_\sigma(0) = -n$.

Lemma 49.4. *The following identity holds,*

$$N_\sigma(w) = \frac{1}{2\pi i}\int_\sigma \frac{1}{z-w}\,dz.$$

Proof. By definition of the contour integral,

$$\begin{aligned}
\int_\sigma \frac{1}{z-w}dz &= \int_a^b \frac{1}{\sigma(t) - w}\dot{\sigma}(t)\,dt \\
&= \int_a^b \frac{1}{r(t)\, e^{i\theta(t)}}\left[\dot{r}(t)\, e^{i\theta(t)} + i\dot{\theta}(t)\, r(t)\, e^{i\theta(t)}\right] dt \\
&= \int_a^b \left[\frac{\dot{r}(t)}{r(t)} + i\dot{\theta}(t)\right] dt = \ln r(t)\,|_a^b + i\Delta\theta = i\Delta\theta.
\end{aligned}$$

Dividing this equation by $2\pi i$ gives the desired result. ∎

Corollary 49.5. *The function $N_\sigma(w)$ is constant as a function of w in each connected component of $\mathbb{C} \setminus \sigma([a,b])$.*

Proof. We have

$$N'_\sigma(w) = \frac{1}{2\pi i}\int_\sigma (z-w)^{-2}\,dz = \frac{-1}{2\pi i}(z-w)^{-1}\,|_{z=\sigma(a)}^{z=\sigma(b)} = 0.$$

∎

Corollary 49.6. *If σ and τ are loops in $\mathbb{C} \setminus \{0\}$ which are homotopic inside of $\mathbb{C} \setminus \{w\}$, then $N_\sigma(w) = N_\tau(w)$. In particular, if σ is homotopic to a constant loop inside of $\mathbb{C} \setminus \{0\}$, then $N_\sigma(w) = 0$.*

Proof. This is the Cauchy-Goursat theorem again. ∎

Corollary 49.7. *If σ is a loop in $\mathbb{C}$, then $N_\sigma(w) = 0$ when w is in the* ***unbounded*** *component of $\mathbb{C} \setminus \sigma([a,b])$.*

Proof. If $|w|$ is very large, then the curve σ can be deformed to the constant loop, C, sitting at $0 \in \mathbb{C}$ inside of $\mathbb{C} \setminus \{w\}$ and therefore $N_\sigma(w) = N_C(w) = 0$. Indeed, if for $s \in [0,1]$ we define $\sigma_s(t) := s \cdot \sigma(t)$, then $\sigma_1 = \sigma$ and $\sigma_0(t) = 0$ for all t and $s \to \sigma_s$ is the desired homotopy.

Alternatively we note that for $|w| >> 1$ that

$$|N_\sigma(w)| = \left|\frac{1}{2\pi i}\int_\sigma \frac{1}{z-w}dz\right| \preceq \frac{1}{|w|}\frac{1}{2\pi}\ell(\sigma)$$

where $\ell(\sigma)$ is the length of σ. For $|w|$ very large it will follow that $|N_\sigma(w)| < 1$ and as $N_\sigma(w) \in \mathbb{Z}$ we must have $N_\sigma(w) = 0$. ∎

Theorem 49.8 (Crossing rules). *If σ is a loop in $\mathbb{C}$ and $w, w' \in \mathbb{C} \setminus \sigma([a,b])$ are as shown in Figure 49.1, then*

$$N_\sigma(w) = N_\sigma(w') + 1. \tag{49.2}$$

If the orientation of σ is reversed, then

$$N_\sigma(w) = N_\sigma(w') - 1. \tag{49.3}$$

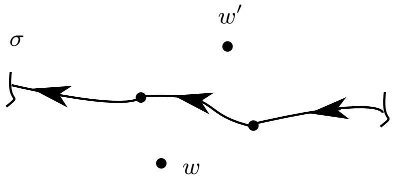


Fig. 49.1. With this configuration we have $N_\sigma(w) = N_\sigma(w') + 1$.

Proof. First decompose σ into the path α followed by γ as indicated on the left side of Figure 49.2 and then let $\tilde{\sigma}$ be the path β followed by γ as shown on the right side of Figure 49.2. Using the properties of winding numbers we have already proved we then have

$$\begin{aligned} N_\sigma(w) &= N_{\tilde{\sigma}}(w) = N_{\tilde{\sigma}}(w') \\ &= \frac{1}{2\pi i}\int_{\tilde{\sigma}} \frac{1}{z-w'}dz = \frac{1}{2\pi i}\left(\int_\beta + \int_\gamma\right)\frac{1}{z-w'}dz \\ &= \frac{1}{2\pi i}\left(\int_\beta - \int_\alpha + \int_\alpha + \int_\gamma\right)\frac{1}{z-w'}dz \\ &= \frac{1}{2\pi i}\left(2\pi i + \int_\sigma \frac{1}{z-w'}dz\right) = 1 + N_\sigma(w') \end{aligned}$$

which proves Eq. (49.2). To prove Eq. (49.3) simply reverse the roles of w' and w in Eq. (49.2). ■

Example 49.9. Using the above rules we may now easily compute the winding numbers of any reasonable closed loop, see for example Figure 49.3.

49.2 Argument Principle Introduction

Theorem 49.10 (Argument Principle). *If $f: D \to \mathbb{C}$ is a meromorphic function and C is a simple closed positively oriented loop in D with having no singularities or zeros on C, then*

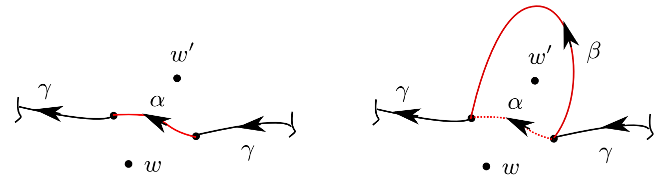


Fig. 49.2. In this figure σ is α followed by γ and $\tilde{\sigma}$ is β followed by γ. Notice that $\tilde{\sigma}$ is homotopic to σ in $\mathbb{C} \setminus \{w\}$ but not in $\mathbb{C} \setminus \{w'\}$.

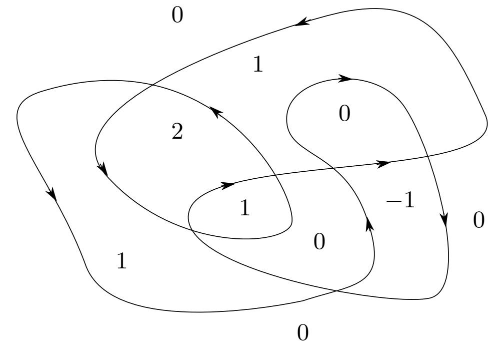
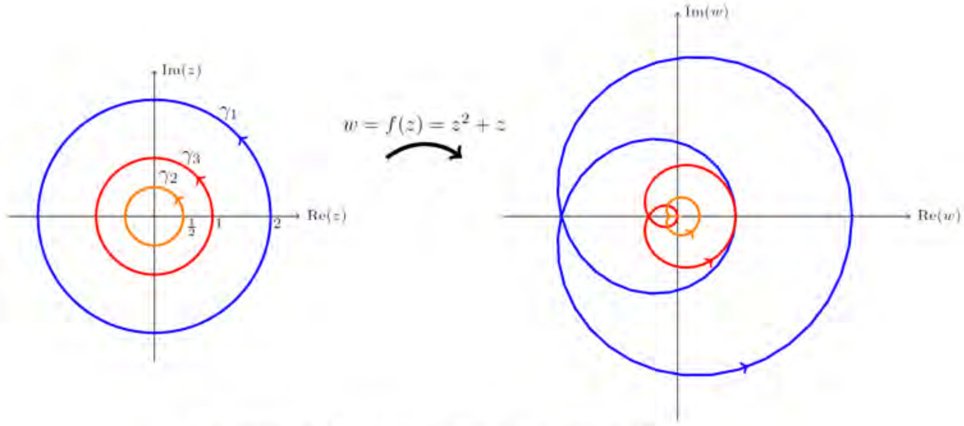


Fig. 49.3. Computing the winding numbers of a closed curve C.

$$\frac{1}{2\pi}\Delta_C \arg(f) := N_{f\circ C}(0) = (\#\text{ of zeros of } f \text{ inside } C) - (\#\text{ of poles of } f \text{ inside } C).$$

The zeros are to be counted with multiplicities and the poles are to be counted with their orders.

The next few figures illustrate the argument principal.



The image of 3 different circles under $f(z) = z^2 + z$.

Let $h(z) = z + z^{-1}$ and note that $h(z) = 0$ iff $z^2 + 1 = 0$, i.e. iff $z = \pm i$.

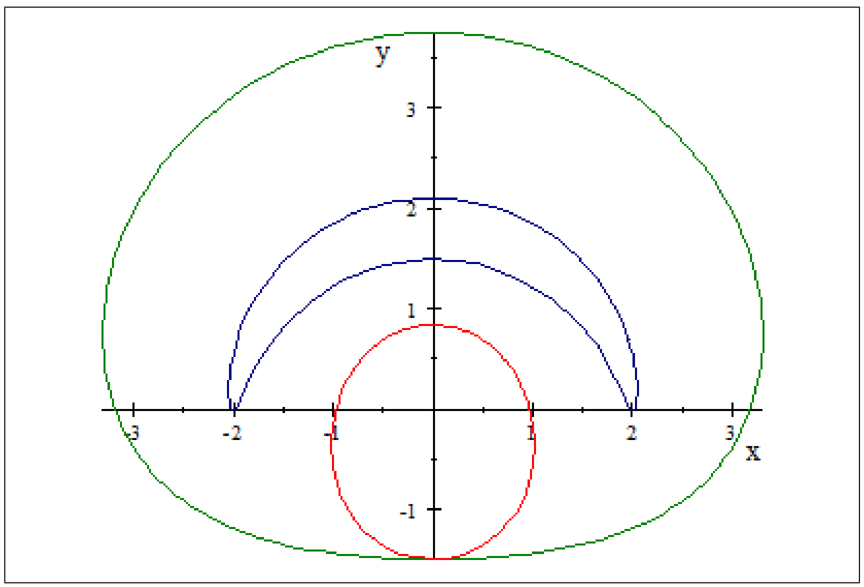


Fig. 49.4. Here $h(z) = z + z^{-1}$ and these are plots of $h\left(i + 3e^{i\theta}\right)$ in green, $h\left(i + 1.5e^{i\theta}\right)$ in blue, and $h\left(i + \frac{1}{2}e^{i\theta}\right)$ in red. All curves go in the counter-clockwise direction.

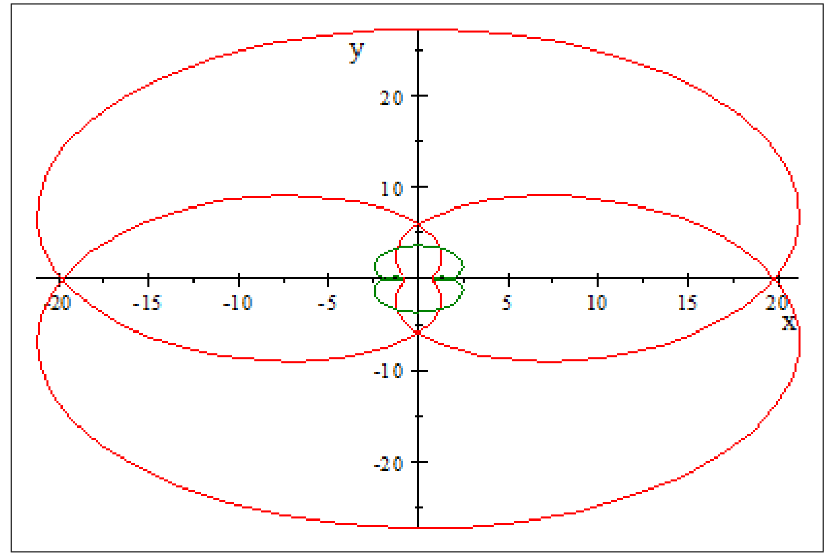


Fig. 49.5. Plots of $\sin\left(4e^{i\theta}\right)$ in red and and $\sin\left(2e^{i\theta}\right)$ in green with winding number 3 and 1 respectively.

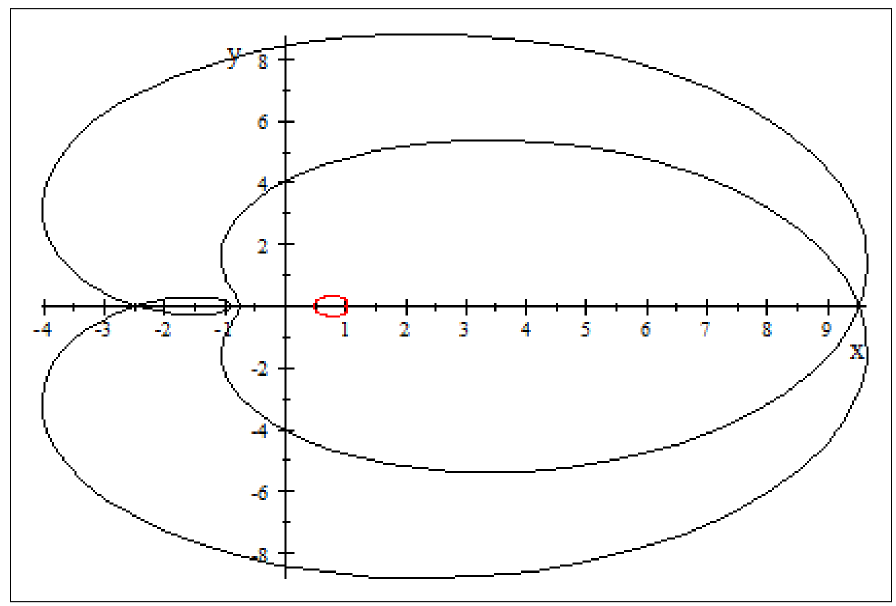


Fig. 49.6. Plots of $\sin\left(1 + 3e^{i\theta}\right)$ in black and $\sin\left(1 + \frac{1}{2}e^{i\theta}\right)$ in red with winding numbers about 0 being 2 and 0 respectively.