Part I

140A Homework Exercises (Fall 2012)

The problems below either come from the lecture notes or from Rudin.

1

HomeWork #1 Due Thursday October 4, 2012

Exercise 1.1. Show that all convergent sequences $\{a_n\}_{n=1}^\infty \subset \mathbb{Q}$ are Cauchy.

Exercise 1.2. Show all Cauchy sequences $\{a_n\}_{n=1}^\infty$ are bounded – i.e. there exists $M \in \mathbb{N}$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Exercise 1.3. Suppose $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are Cauchy sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^\infty$ and $\{a_n \cdot b_n\}_{n=1}^\infty$ are Cauchy.

Exercise 1.4. Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are convergent sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^\infty$ and $\{a_n \cdot b_n\}_{n=1}^\infty$ are convergent in $\mathbb{Q}$ and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \text{ and}$$
$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

Exercise 1.5. Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are convergent sequences in $\mathbb{Q}$ such that $a_n \leq b_n$ for all n. Show $A := \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n =: B$.

Exercise 1.6 (Sandwich Theorem). Assume that $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ are convergent sequences in $\mathbb{Q}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. If $\{x_n\}_{n=1}^\infty$ is another sequence in $\mathbb{Q}$ which satisfies $a_n \leq x_n \leq b_n$ for all n, then

$$\lim_{n\to\infty} x_n = a := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Please note that that main part of the problem is to show that $\lim_{n\to\infty} x_n$ exists in $\mathbb{Q}$. **Hint:** start by showing; if $a \leq x \leq b$ then $|x| \leq \max(|a|, |b|)$.

Exercise 1.7. Use the following outline to construct another Cauchy sequence $\{q_n\}_{n=1}^\infty \subset \mathbb{Q}$ which is **not** convergent in $\mathbb{Q}$.

1. Recall that there is no element $q \in \mathbb{Q}$ such that $q^2 = 2$. To each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be chosen so that
$$\frac{m_n^2}{n^2} < 2 < \frac{(m_n + 1)^2}{n^2} \tag{1.1}$$
and let $q_n := \frac{m_n}{n}$.
2. Verify that $q_n^2 \to 2$ as $n \to \infty$ and that $\{q_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{Q}$.
3. Show $\{q_n\}_{n=1}^\infty$ does not have a limit in $\mathbb{Q}$.

2

HomeWork #2 Due Thursday October 11, 2012

Problems from Rudin:
Chapter 1: 1.1, 1.4.
Chapter 3: 3.2

Definition 2.1 (Subsequence). *We say a sequence,* $\{y_k\}_{k=1}^{\infty}$ *is a* ***subsequence*** *of another sequence,* $\{x_n\}_{n=1}^{\infty}$, *provided there exists a strictly increasing function,* $\mathbb{N} \ni k \to n_k \in \mathbb{N}$ *such that* $y_k = x_{n_k}$ *for all* $k \in \mathbb{N}$.

Exercise 2.1. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$ which has a convergent subsequence, $\{y_k = x_{n_k}\}_{k=1}^{\infty}$. Show that $\lim_{n\to\infty} x_n$ exists and is equal to $\lim_{k\to\infty} y_k$.

Exercise 2.2. Suppose that $\alpha \subset \mathbb{Q}$ is a cut as in Definition 2.27. Show α is bounded from above. Then let $m := \sup \alpha$ and show that $\alpha = \alpha_m$, where α_m

$$\alpha_m := \{y \in \mathbb{Q} : y < m\}.$$

Also verify that α_m is a cut for all $m \in \mathbb{R}$. [In this way we see that we may identify $\mathbb{R}$ with the cuts of $\mathbb{Q}$. This motivated Dedekind's construction of the real numbers as described in Rudin.]

Exercise 2.3 (Do not hand in). Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $A := \lim_{n\to\infty} a_n$ and $B := \lim_{n\to\infty} b_n$ exists in $\mathbb{R}$. Then;

1. $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
2. $\lim_{n\to\infty} |a_n| = |A|$.
3. If $A \neq 0$ then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.
4. $\lim_{n\to\infty} (a_n + b_n) = A + B$.
5. $\lim_{n\to\infty} (a_n b_n) = A \cdot B$.
6. If $a_n \leq b_n$ for all n, then $A \leq B$.
7. If $\{x_n\} \subset \mathbb{R}$ is another sequence such that $a_n \leq x_n \leq b_n$ and $A = B$, then $\lim_{n\to\infty} x_n = A = B$.

[The point here is to convince yourself that the proofs of the analogous statement you gave when everything was in $\mathbb{Q}$ still hold true here.]