Part I

140A Homework Exercises (Fall 2012)

- The problems below either come from the lecture notes or from Rudin's "Principles of Real Analysis," 3rd ed.
- Although pictures are not typically provided in these solutions, I would highly advise that the reader draw appropriate pictures whenever possible. Many of the proofs to follow are really a transcription of a very natural picture proof to a formal written proof.

1

HomeWork #1 Due Thursday October 4, 2012

Exercise 1.1. Show that all convergent sequences $\{a_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ are Cauchy.

Exercise 1.2. Show all Cauchy sequences $\{a_n\}_{n=1}^{\infty}$ are bounded – i.e. there exists $M \in \mathbb{N}$ such that

$$|a_n| \leq M \text{ for all } n \in \mathbb{N}.$$

Exercise 1.3. Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^{\infty}$ and $\{a_n \cdot b_n\}_{n=1}^{\infty}$ are Cauchy.

Exercise 1.4. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$. Show $\{a_n + b_n\}_{n=1}^{\infty}$ and $\{a_n \cdot b_n\}_{n=1}^{\infty}$ are convergent in $\mathbb{Q}$ and

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n \text{ and}$$
$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n.$$

Exercise 1.5. Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$ such that $a_n \leq b_n$ for all n. Show $A := \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n =: B$.

Exercise 1.6 (Sandwich Theorem). Assume that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences in $\mathbb{Q}$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. If $\{x_n\}_{n=1}^{\infty}$ is another sequence in $\mathbb{Q}$ which satisfies $a_n \leq x_n \leq b_n$ for all n, then

$$\lim_{n\to\infty} x_n = a := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Please note that that main part of the problem is to show that $\lim_{n\to\infty} x_n$ exists in $\mathbb{Q}$. **Hint:** start by showing; if $a \leq x \leq b$ then $|x| \leq \max(|a|, |b|)$.

Exercise 1.7. Use the following outline to construct another Cauchy sequence $\{q_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ which is **not** convergent in $\mathbb{Q}$.

1. Recall that there is no element $q \in \mathbb{Q}$ such that $q^2 = 2$. To each $n \in \mathbb{N}$ let $m_n \in \mathbb{N}$ be chosen so that

$$\frac{m_n^2}{n^2} < 2 < \frac{(m_n + 1)^2}{n^2} \tag{1.1}$$

   and let $q_n := \frac{m_n}{n}$.
2. Verify that $q_n^2 \to 2$ as $n \to \infty$ and that $\{q_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$.
3. Show $\{q_n\}_{n=1}^{\infty}$ does not have a limit in $\mathbb{Q}$.

2

HomeWork #2 Due Thursday October 11, 2012

Problems from Rudin:

Chapter 1: 1.1, 1.4.

Chapter 3: 3.2

Note: this problem may seem trivial, but the point is to realize that a least upper bound for E need not actually be an element of E!

Definition 2.1 (Subsequence). *We say a sequence, $\{y_k\}_{k=1}^{\infty}$ is a **subsequence** of another sequence, $\{x_n\}_{n=1}^{\infty}$, provided there exists a strictly increasing function, $\mathbb{N} \ni k \to n_k \in \mathbb{N}$ such that $y_k = x_{n_k}$ for all $k \in \mathbb{N}$. [Example, $n_k = k^2 + 3$, and $\{y_k := x_{k^2+3}\}_{k=1}^{\infty}$ would be a subsequence of $\{x_n\}_{n=1}^{\infty}$.]*

Exercise 2.1. Suppose that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}$ (or $\mathbb{R}$) which has a convergent subsequence, $\{y_k = x_{n_k}\}_{k=1}^{\infty}$ in $\mathbb{Q}$ (or $\mathbb{R}$). Show that $\lim_{n\to\infty} x_n$ exists and is equal to $\lim_{k\to\infty} y_k$.

Exercise 2.2. Suppose that $\alpha \subset \mathbb{Q}$ is a cut as in Definition 2.27. Show α is bounded from above. Then let $m := \sup \alpha$ and show that $\alpha = \alpha_m$, where α_m

$$\alpha_m := \{y \in \mathbb{Q} : y < m\}.$$

Also verify that α_m is a cut for all $m \in \mathbb{R}$. [In this way we see that we may identify $\mathbb{R}$ with the cuts of $\mathbb{Q}$. This should motivate Dedekind's construction of the real numbers as described in Rudin.]

Exercise 2.3 (Do not hand in). Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $A := \lim_{n\to\infty} a_n$ and $B := \lim_{n\to\infty} b_n$ exists in $\mathbb{R}$. Then;

1. $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.
2. $\lim_{n\to\infty} |a_n| = |A|$.
3. If $A \neq 0$ then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$.
4. $\lim_{n\to\infty} (a_n + b_n) = A + B$.
5. $\lim_{n\to\infty} (a_n b_n) = A \cdot B$.
6. If $a_n \leq b_n$ for all n, then $A \leq B$.
7. If $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is another sequence such that $a_n \leq x_n \leq b_n$ and $A = B$, then $\lim_{n\to\infty} x_n = A = B$.

[The point here is to convince yourself that the proofs of the analogous statement you gave when everything was in $\mathbb{Q}$ still hold true here. This would be a good time to make sure that you understand these proofs!]

3

HomeWork #3 Due Thursday October 17, 2012

Problems from Rudin:

Chapter 1: 1.6 (Hint: for part d) use the Sup-Sup theorem or its corollary.)

Chapter 3: 3.3, 3.4. (Look at but do not hand in. Hint: derive formulas for s_{2n} and s_{2n+1}.)

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0, \quad s_{2m} = \frac{s_{2m-1}}{2}, \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution: If we write out the first few terms we obtain

$$\begin{aligned} s_1 &= 0 \\ s_2 &= 0 \\ s_3 &= \frac{1}{2} \\ s_4 &= \frac{1}{4} \\ s_5 &= \frac{1}{2} + \frac{1}{4} \\ s_6 &= \frac{1}{4} + \frac{1}{8} \\ s_7 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \\ s_8 &= \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \end{aligned}$$

thus we are lead to write

$$\begin{aligned} s_{2m+1} &= \sum_{k=1}^{m} \left(\frac{1}{2}\right)^k \quad m \geq 1 \\ s_{2m} &= \sum_{k=2}^{m} \left(\frac{1}{2}\right)^k \quad m \geq 2. \end{aligned} \tag{4.1}$$

To prove this we proceed by induction on m. For $m = 1$ we have that $s_3 = 1/2 = \sum_{k=1}^{1}(1/2)^k$. For $m = 2$ we have that $s_5 = 1/2 + 1/4 = \sum_{k=1}^{2}(1/2)^k$ and $s_4 = 1/4 = \sum_{k=2}^{2}(1/2)^k$. Now,

Exercise 3.1. Let $\{a_n\}_{n=1}^{\infty}$ be the sequence given by,

$$(-1, 2, 3, -1, 2, 3, -1, 2, 3, \dots).$$

Find $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Exercise 3.2. Show $\liminf_{n\to\infty}(-a_n) = -\limsup_{n\to\infty} a_n$.

Exercise 3.3 (Do not hand in). If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences such that $a_n \leq b_n$ for a.a. n, then

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n \text{ and } \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n. \tag{3.1}$$

Exercise 3.4. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in $\mathbb{R}$. Show

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n \tag{3.2}$$

provided that the right side of Eq. (3.8) is well defined, i.e. no $\infty - \infty$ or $-\infty + \infty$ type expressions. (It is OK to have $\infty + \infty = \infty$ or $-\infty - \infty = -\infty$, etc.)

Exercise 3.5. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in $[0, \infty) \subset \mathbb{R}$. Show

$$\limsup_{n\to\infty}(a_n b_n) \leq \limsup_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n, \tag{3.3}$$

provided the right hand side of (3.9) is not of the form $0 \cdot \infty$ or $\infty \cdot 0$.

3.1 Test #1, Monday 10/22/2012

suppose inductively that Equation (4.1) holds for $m \geq 2$. Then,

$$s_{2(m+1)+1} = \frac{1}{2} + s_{2(m+1)} = \frac{1}{2} + \frac{s_{2m+1}}{2} = \frac{1}{2} + \sum_{k=1}^{m} \left(\frac{1}{2}\right)^{k+1} = \sum_{k=1}^{m+1} \left(\frac{1}{2}\right)^{k}$$

$$s_{2(m+1)} = \frac{s_{2m+1}}{2} = \frac{1}{2} \sum_{k=1}^{m} \left(\frac{1}{2}\right)^{k} = \sum_{k=1}^{m} \left(\frac{1}{2}\right)^{k+1} = \sum_{k=2}^{m+1} \left(\frac{1}{2}\right)^{k}$$

and so Equation (4.1) holds. We thus see that there are only two subsequential limits

$$\lim_{m\to\infty} s_{2m} = \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k} = \frac{1}{2} \tag{4.2}$$

$$\lim_{m\to\infty} s_{2m+1} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k} = 1 \tag{4.3}$$

where the two limiting values are given by Theorem 3.26 on page 61 of the text (geometric series). These are the only possible limits because if we have a subsequence $\{s_{n_k}\}$ then either $\{n_k\}$ has infinitely many evens, infinitely many odds, or infinitely many of both. If there are infinitely many of both even and odd numbers, then $\{s_{n_k}\}$ wouldn't be Cauchy sequence because for all $N \geq 1$ we could find $n_{k_1}, n_{k_2} \geq N$ such that n_{k_1} was even and n_{k_2} was odd and then we would have that $|s_{n_{k_1}} - s_{n_{k_2}}| \geq 1/2$. If there are only finitely many odds in $\{n_k\}$ then we would have that $\lim_{k\to\infty} s_{n_k} = \lim_{m\to\infty} s_{2m}$. Specifically, letting $N > \max_k\{n_k \text{ odd }\}$ we see that $|s_{n_k} - s_{n_j}| = \sum_{k=n_j}^{n_k} (1/2)^k$ for all $n_k \geq n_j \geq N$ precisely because both n_j and n_k are even. Since $\sum_{k=2}^{\infty} (1/2)^k$ converges, it is a Cauchy sequence by Theorem 3.11 on page 53 of the text and so $\sum_{k=n_j}^{n_k} (1/2)^k$ can be made arbitrarily small for large enough n_j and n_k. If there are only finitely many evens in $\{n_k\}$ then we would have that $\lim_{k\to\infty} s_{n_k} = \lim_{m\to\infty} s_{2m+1}$. Specifically, letting $N > \max_k\{n_k \text{ even }\}$ we see that $|s_{n_k} - s_{n_j}| = \sum_{k=n_j}^{n_k} (1/2)^k$ for all $n_k \geq n_j \geq N$ precisely because both n_j and n_k are odd. Since $\sum_{k=1}^{\infty} (1/2)^k$ converges, it is a Cauchy sequence by Theorem 3.11 on page 53 of the text and so $\sum_{k=n_j}^{n_k} (1/2)^k$ can be made arbitrarily small for large enough n_j and n_k. Since Equations (4.2) and (4.3) are the only two subsequential limits we find that the upper and lower limits are

$$\liminf_{m\to\infty} s_m = \frac{1}{2}$$
$$\limsup_{m\to\infty} s_m = 1.$$

4

HomeWork #4 Due Thursday October 25, 2012

You should do these problems but they will **not** be collected this week.

Problems from Rudin:

Chapter 1: 1.11, 1.12, 1.13, 1.17 [This will appear on the next homework.]

Exercise 4.1. Let $(\mathbb{F}, P)$ be an ordered field and $x, y \in \mathbb{F}$ with $y > x$. Show;

1. $y + a > x + a$ for all $a \in \mathbb{F}$,
2. $-x > -y$,
3. if we further suppose $x > 0$, show $\frac{1}{x} > \frac{1}{y}$.

Exercise 4.2. Show $\lim_{n\to\infty} \alpha^n = \infty$ and $\lim_{n\to\infty} \frac{1}{\alpha^n} = 0$ whenever $\alpha > 1$.

Exercise 4.3. Suppose $\alpha > 1$ and $k \in \mathbb{N}$, show there is a constant $c = c(\alpha, k) > 0$ such that $\alpha^n \geq cn^k$ for all $n \in \mathbb{N}$. [In words, α^n grows in n faster than any polynomial in n.]

Exercise 4.4. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and let

$$A := \{y \in \mathbb{R} : a_n \geq y \text{ for a.a. } n\}.$$

Then $\sup A = \liminf_{n\to\infty} a_n$ with the convention that $\sup A = -\infty$ if $A = \emptyset$.

Exercise 4.5. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Show $\limsup_{n\to\infty} a_n = a^* \in \mathbb{R}$ iff for all $\varepsilon > 0$,

$$\begin{aligned} &a_n \leq a^* + \varepsilon \text{ for a.a. } n. \text{ and} \\ &a^* - \varepsilon \leq a_n \text{ i.o. } n. \end{aligned}$$

Similarly, show $\liminf_{n\to\infty} a_n = a_* \in \mathbb{R}$ iff for all $\varepsilon > 0$,

$$\begin{aligned} &a_n \leq a_* + \varepsilon \text{ i.o. } n. \text{ and} \\ &a_* - \varepsilon \leq a_n \text{ for a.a. } n. \end{aligned}$$

Notice that this exercise gives another proof of item 2. of Proposition 3.28 in the case all limits are real valued.

Exercise 4.6 (Cauchy Complete $\implies$ L.U.B. Property). Suppose that $\mathbb{R}$ denotes any ordered field which is Cauchy complete and $\mathbb{N}$ is not bounded from above. Show $\mathbb{R}$ has the least upper bound property and therefore is the field of real numbers.

5

HomeWork #5 Due Thursday November 1, 2012

Problems from Rudin:
Chapter 1: 1.11, 1.12, 1.13, 1.17(1.17 is not to be handed in.)

Exercise 5.1. Let X be a set and for any function $f : X \to \mathbb{C}$, let

$$\|f\|_u := \sup_{x \in X} |f(x)|.$$

Show $Z := \{f : X \to \mathbb{C} : \|f\|_u < \infty\}$ is a vector space and $\|\cdot\|_u$ is a norm on Z.

For the next two exercise you will be using some concepts from calculus which we will developed in detail next quarter. For now, I assume you know what the Riemann integral is for continuous functions on $[0,1]$ with values in $\mathbb{R}$. Let Z denote the continuous functions[1] on $[0,1]$ with values in $\mathbb{R}$. The only properties that you need to know about the Riemann integral are;

1. The integral is linear, namely for all $f, g \in Z$ and $\lambda \in \mathbb{R}$,
$$\int_0^1 (f(t) + \lambda g(t))dt = \int_0^1 f(t)\,dt + \lambda \int_0^1 g(t)\,dt.$$
2. If $f, g \in Z$ and $f(t) \le g(t)$ for all $t \in [0,1]$, then
$$\int_0^1 f(t)\,dt \le \int_0^1 g(t)\,dt.$$
3. For all $f \in Z$,
$$\left|\int_0^1 f(t)\,dt\right| \le \int_0^1 |f(t)|\,dt.$$
In fact this item follows from items 1. and 2. Indeed, since $\pm f(t) \le |f(t)|$ for all t, we find
$$\pm \int_0^1 f(t)\,dt = \int_0^1 \pm f(t)\,dt \le \int_0^1 |f(t)|\,dt \iff \left|\int_0^1 f(t)\,dt\right| \le \int_0^1 |f(t)|\,dt.$$

Exercise 5.2. Let Z denote the continuous function on $[0,1]$ with values in $\mathbb{R}$ and for $f \in Z$ let

$$\|f\|_1 := \int_0^1 |f(t)|\,dt.$$

Show $\|\cdot\|_1$ satisfies,

1. (Homogeneity) $\|\lambda f\| = |\lambda|\,\|f\|$ for all $\lambda \in \mathbb{R}$ and $f \in Z$.

[1] The notion of continuity will be formally developed shortly.

2. (Triangle inequality) $\|f+g\| \leq \|f\| + \|g\|$ for all $f, g \in Z$.

Exercise 5.3. Let Z denote the continuous function on $[0,1]$ with values in $\mathbb{R}$ and for $f \in Z$ let

$$\|f\|_2 := \sqrt{\int_0^1 |f(t)|^2 \, dt}.$$

Show;

1. for $f, g \in Z$ that
$$\left| \int_0^1 f(t) \, g(t) \, dt \right| \leq \|f\|_2 \cdot \|g\|_2 ,$$
2. Homogeneity) $\|\lambda f\|_2 = |\lambda| \, \|f\|_2$ for all $\lambda \in \mathbb{R}$ and $f \in Z$, and
3. (Triangle inequality) $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$ for all $f, g \in Z$.

For the remaining exercises, let X and Y be sets, $f : X \to Y$ be a function and $\{A_i\}_{i \in I}$ be an indexed family of subsets of Y, verify the following assertions.

Exercise 5.4. $(\cap_{i \in I} A_i)^c = \cup_{i \in I} A_i^c$.

Exercise 5.5. Suppose that $B \subset Y$, show that $B \setminus (\cup_{i \in I} A_i) = \cap_{i \in I} (B \setminus A_i)$.

Exercise 5.6. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$.

Exercise 5.7. $f^{-1}(\cap_{i \in I} A_i) = \cap_{i \in I} f^{-1}(A_i)$.

Exercise 5.8. Find a counterexample which shows that $f(C \cap D) = f(C) \cap f(D)$ need not hold.

6

HomeWork #6 Due Thursday November 8, 2012

Exercise 6.1. Show that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$.

Exercise 6.2. Let $\mathbb{Q}[t]$ be the set of polynomial functions, p, such that p has rationale coefficients. That is $p \in \mathbb{Q}[t]$ iff there exists $n \in \mathbb{N}_0$ and $a_k \in \mathbb{Q}$ for $0 \leq k \leq n$ such that

$$p(t) = \sum_{k=0}^{n} a_k t^k \text{ for all } t \in \mathbb{R}. \tag{6.1}$$

Show $\mathbb{Q}[t]$ is a countable set.

Definition 6.1 (Algebraic Numbers). *A real number, $x \in \mathbb{R}$, is called **algebraic number**, if there is a non-zero polynomial $p \in \mathbb{Q}[t]$ such that $p(x) = 0$. [That is to say, $x \in \mathbb{R}$ is algebraic if it is the root of a non-zero polynomial with coefficients from $\mathbb{Q}$.]*

Note that for all $q \in \mathbb{Q}$, $p(t) := t - q$ satisfies $p(q) = 0$. Hence all rational numbers are algebraic. But there are many more algebraic numbers, for example $y^{1/n}$ is algebraic for all $y \geq 0$ and $n \in \mathbb{N}$.

Exercise 6.3. Show that the set of algebraic numbers is countable. [Hint: any polynomial of degree n has at most n – real roots.] In particular, "most" irrational numbers are not algebraic numbers, i.e. there is still an uncountable number of non-algebraic numbers..

Exercise 6.4. Show that convergent sequences in metric spaces are always Cauchy sequences. The converse is not always true. For example, let $X = \mathbb{Q}$ be the set of rational numbers and $d(x, y) = |x - y|$. Choose a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{Q}$ which converges to $\sqrt{2} \in \mathbb{R}$, then $\{x_n\}_{n=1}^{\infty}$ is $(\mathbb{Q}, d)$ – Cauchy but not $(\mathbb{Q}, d)$ – convergent. Of course the sequence is convergent in $\mathbb{R}$..

Exercise 6.5. If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d), then $\lim_{n\to\infty} d(x_n, y)$ exists in $\mathbb{R}$ for all $y \in X$. In particular, $\{d(x_n, y)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ for all $y \in X$..

Exercise 6.6. Let X be a set and (Y, ρ) be a complete metric space. Suppose that $f_n : X \to Y$ are functions such that

$$\delta_{m,n} := \sup_{x \in X} d(f_n(x), f_m(x)) \to 0 \text{ as } m, n \to \infty.$$

Show there exists a (unique) functions, $f : X \to Y$ such that

$$\lim_{n\to\infty} \sup_{x \in X} d(f_n(x), f(x)) = 0.$$

Hint: mimic the proof of Lemma 6.22.

Exercise 6.7. Suppose that (X, d) is a metric space and $f, g : X \to \mathbb{C}$ are two continuous functions on X. Show;

1. $f + g$ is continuous,

2. $f \cdot g$ is continuous,
3. f/g is continuous provided $g(x) \neq 0$ for all $x \in X$.

Exercise 6.8. Show the following functions from $\mathbb{C}$ to $\mathbb{C}$ are continuous.

1. $f(z) = c$ for all $z \in \mathbb{C}$ where $c \in \mathbb{C}$ is a constant.
2. $f(z) = |z|$.
3. $f(z) = z$ and $f(z) = \bar{z}$.
4. $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$.
5. $f(z) = \sum_{m,n=0}^{N} a_{m,n} z^m \bar{z}^n$ where $a_{m,n} \in \mathbb{C}$.

Exercise 6.9. Suppose now that (X, ρ), (Y, d), and (Z, δ) are three metric spaces and $f : X \to Y$ and $g : Y \to Z$. Let $x \in X$ and $y = f(x) \in Y$, show $g \circ f : X \to Z$ is continuous at x if f is continuous at x and g is continuous at y. Recall that $(g \circ f)(x) := g(f(x))$ for all $x \in X$. In particular this implies that if f is continuous on X and g is continuous on Y then $f \circ g$ is continuous on X.

Exercise 6.10 (Intermediate value theorem). Suppose that $-\infty < a < b < \infty$ and $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a) \leq f(b)$. Show for any $y \in [f(a), f(b)]$, there exists a $c \in [a, b]$ such that $f(c) = y$.[1] **Hint:** Let $S := \{t \in [a, b] : f(t) \leq y\}$ and let $c := \sup(S)$.

[1] The same result holds for $y \in [f(b), f(a)]$ if $f(b) \leq f(a)$ – just replace f by $-f$ in this case.

7

HomeWork #7 Due Thursday November 15, 2012

Exercise 7.1. Let $f:[a,b]\to[c,d]$ be a strictly increasing (i.e. $f(x_1)<f(x_2)$ whenever $x_1<x_2$) continuous function such that $f(a)=c$ and $f(b)=d$. Then f is bijective and the inverse function, $g:=f^{-1}:[c,d]\to[a,b]$, is strictly increasing and is continuous.

Exercise 7.2. Let $\|\cdot\|_1$, $\|\cdot\|_u$, and $\|\cdot\|_2$ be the three norms on $\mathbb{C}^n$ given above. Show for all $z\in\mathbb{C}^n$ that

$$\begin{aligned}
&\|z\|_u \le \|z\|_1 \le n\|z\|_u,\\
&\|z\|_1 \le \sqrt{n}\|z\|_2, (\textbf{Hint}\text{: use Cauchy Schwarz.})\\
&\|z\|_2 \le \sqrt{\|z\|_u\cdot\|z\|_1} \le \|z\|_1 \text{ and}\\
&\|z\|_2 \le \sqrt{n}\|z\|_u.
\end{aligned}$$

It follows from these inequalities that $\|\cdot\|_1$, $\|\cdot\|_u$, and $\|\cdot\|_2$ are equivalent norms on $\mathbb{C}^n$.

Exercise 7.3. Let Z denote the continuous function on $[0,1]$ with values in $\mathbb{R}$ and as above let

$$\|f\|_1:=\int_0^1|f(t)|\,dt,\quad \|f\|_2:=\sqrt{\int_0^1|f(t)|^2\,dt},\text{ and }\|f\|_u=\sup_{0\le t\le1}|f(t)|.$$

Show for all $f\in Z$ that;

$$\|f\|_1\le\|f\|_2 \quad\text{and}\quad \|f\|_2\le\|f\|_u.$$

[**Hint:** for the first inequality use Cauchy Schwarz.] Also show there is no constant $C<\infty$ such that

$$\|f\|_u\le C\|f\|_2 \text{ for all } f\in Z. \tag{7.1}$$

[**Hint:** consider the sequence, $f_n(t)=t^n$.]

Exercise 7.4 (Continuity of integration). Let $Z=C([0,1],\mathbb{R})$ be the continuous functions from $[0,1]$ to $\mathbb{R}$ and $\|\cdot\|_u$ be the uniform norm, $\|f\|_u:=\sup_{0\le t\le1}|f(t)|$. Define $K:Z\to Z$ by

$$K(f)(x):=\int_0^x f(t)\,dt \text{ for all } x\in[0,1].$$

Show that K is a Lipschitz function. In more detail, show

$$\|K(f)-K(g)\|_u\le\|f-g\|_u \text{ for all } f,g\in Z.$$

In this problem please take for granted the standard properties of the integral including

1. The function $x\to K(f)(x)$ is indeed continuous (in fact differentiable by the fundamental theorem of calculus).
2. $K:Z\to Z$ is a linear transformation.

3. If $f(t) \leq g(t)$ for all $t \in [0,1]$, then $\int_0^x f(t)\,dt \leq \int_0^x g(t)\,dt$ for all $x \in X$.
4. From 3. it follows that $\left|\int_0^x f(t)\,dt\right| \leq \int_0^x |f(t)|\,dt$.

Exercise 7.5 (Discontinuity of differentiation). Let Z be the polynomial functions in $C([0,1],\mathbb{R})$, i.e. functions of the form $p(t) = \sum_{k=0}^n a_k t^k$ with $a_k \in \mathbb{R}$. As above we let $\|p\|_u := \sup_{0\leq t\leq 1} |p(t)|$. Define $D : Z \to Z$ by $D(p) = p'$, i.e. if $p(t) = \sum_{k=0}^n a_k t^k$ then

$$D(p)(t) = \sum_{k=1}^n k a_k t^{k-1}.$$

1. Show D is discontinuous at $\mathbf{0}$ – where $\mathbf{0}$ represents the zero polynomial.
2. Show D is discontinuous at all points $p \in Z$.

Exercise 7.6. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for $x \in [0,1]$. Show $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in [0,1]$ and find f explicitly. Show that f_n does **not** converge to f uniformly.

Definition 7.1. *Let (X,d) be a metric space. We say $A \subset X$ is **dense** in X if for all $x \in X$, there exists $\{x_n\}_{n=1}^\infty \subset A$ such that $x = \lim_{n\to\infty} x_n$. [In words, all points in X are limit points of sequences in A.] A metric space is said to be **separable** if it contains a countable dense subset, D.*

Exercise 7.7. Suppose that (X,d) is a separable metric space and Y is a non-empty subset of X which is also a metric space by restricting d to Y. Show (Y,d) is separable.

Exercise 7.8. Let $n \in \mathbb{N}$. Show any non-empty subset $Y \subset \mathbb{C}^n$ equipped with the metric,

$$d(x,y) = \|y - x\| \text{ for all } x, y \in Y$$

is separable, where $\|\cdot\|$ is either $\|\cdot\|_u$, $\|\cdot\|_1$, or $\|\cdot\|_2$.