Part I

140B Homework Exercises (Winter 2013)

- The problems below either come from the lecture notes or from Rudin's "Principles of Real Analysis," 3rd ed.
- Although pictures are not typically provided in these solutions, I would highly advise that the reader draw appropriate pictures whenever possible. Many of the proofs to follow are really a transcription of a very natural picture proof to a formal written proof.

1

HomeWork #1 Due Friday, January 11, 2013)

Exercise 1.1. Prove item 2. of Theorem 9.7. If $\{C_\alpha\}_{\alpha\in I}$ is a collection of closed subsets of X, then $\cap_{\alpha\in I}C_\alpha$ is closed in X.

Exercise 1.2. Give an example of a collection of closed subsets, $\{A_n\}_{n=1}^{\infty}$, of $\mathbb{C}$ such that $\cup_{n=1}^{\infty}A_n$ is not closed.

Exercise 1.3. Prove Corollary 9.8 from the text. That is; if (X,d) be a metric space, then the collection of open subsets, τ_d, of X satisfy;

1. X and $\emptyset$ are in τ_d.
2. τ_d is closed under taking arbitrary unions. i.e. if $\{U_\alpha\}_{\alpha\in I}$ is a collection of open sets then $\cup_{\alpha\in I}U_\alpha$ is open..
3. τ_d is closed under taking finite intersections, i.e. if U and V are open sets then then $U\cap V$ is open as well.

Exercise 1.4. Let U be a subset of a metric space (X,d). Show the following are equivalent;

1. U is open,
2. for all $z\in U$ there exists $\rho>0$ such that $B_z(\rho)\subset U$.
3. U can be written as a union of open balls.

Exercise 1.5. Let (X,d) be a metric space and $\{x_1,\dots,x_n\}$ be a finite subset of X. Show $X\setminus\{x_1,\dots,x_n\}$ is an open subset.

Exercise 1.6. Let (X,d) be a complete metric space. Let $A\subset X$ be a subset of X viewed as a metric space using $d|_{A\times A}$. Show that $(A,d|_{A\times A})$ is complete iff A is a closed subset of X.

Exercise 1.7. If A is a non-empty subset of X, then $d_A=d_{\bar{A}}$.

Exercise 1.8. Given $A\subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A}=\cap\{F:A\subset F\subset X \text{ with } F \text{ closed}\}. \tag{1.1}$$

That is to say $\bar{A}$ is the smallest closed set containing A.

2

HomeWork #2 Due Friday, January 18, 2013)

Exercise 2.1. Let $(X, \|\cdot\|)$ be a normed space and $d(x,y) := \|y-x\|$. Give a formal proof that;

1. $\overline{C_x(\rho)}^o = B_x(\rho)$,
2. $\overline{B_x(\rho)} = C_x(\rho)$,
3. $\mathrm{bd}(C_x(\rho)) = \mathrm{bd}(B_x(\rho)) = \{y \in X : \|y-x\| = \rho\}$.

Exercise 2.2. Suppose that A and B are subsets of a metric space, show $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Exercise 2.3. Given an example showing that $\overline{\cup_{n=1}^{\infty} A_n}$ need not be equal to $\cup_{n=1}^{\infty} \overline{A_n}$.

Exercise 2.4 (Extreme value theorem). Let K be sequentially compact subset of X and $f : K \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf_{x\in K} f(x) \leq \sup_{x\in K} f(x) < \infty$ and there exists $a, b \in K$ such that $f(a) = \inf_{x\in K} f(x)$ and $f(b) = \sup_{x\in K} f(x)$. **Hint:** first argue that there exists $\{z_n\}_{n=1}^{\infty} \subset K$ such that $f(z_n) \uparrow \sup_{x\in K} f(x)$ as $n \to \infty$.

Exercise 2.5 (Uniform Continuity). Let K be sequentially compact subset of X and $f : K \to \mathbb{C}$ be a continuous function. Show that f is uniformly continuous, i.e. if $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ if $w, z \in K$ with $d(w,z) < \delta$. **Hint:** prove the contrapositive.

Exercise 2.6. If (X,d) is a metric space and suppose that $K \subset X$ is sequentially compact. Show that closed subsets, $C \subset K$, of K are sequentially compact as well.

Exercise 2.7. If $K \subset \mathbb{R}$ is sequentially compact then $\sup(K) \in K$, i.e. $\sup(K) = \max(K)$.

Exercise 2.8. Let (X,d) and (Y,ρ) be metric spaces, $K \subset X$ be a sequentially compact set, and $f : X \to Y$ be a continuous function. Show $f(K)$ is sequentially compact in Y. In particular, for $C \subset K$ closed, we have $f(C)$ is closed and in fact sequentially compact in Y.

Exercise 2.9 (Rudin 2.14). Give an example of open cover of $(0,1)$ which has no finite subcover.

Exercise 2.10. Suppose $f : X \to Y$ is continuous and $K \subset X$ is open cover compact, then $f(K)$ is a open cover compact subset of Y. Give an example of continuous map, $f : X \to Y$, and a compact subset K of Y such that $f^{-1}(K)$ is not compact.

3

HomeWork #3 Due Friday, January 25, 2013)

Exercise 3.1 (Extreme value theorem II). Suppose that $f : X \to \mathbb{R}$ is a continuous function on a metric space (X, d). Further assume there exists a sequentially compact subset $K \subset X$ and $x_0 \in K$ such that $f(x_0) \leq f(x)$ for all $x \in X \setminus K$. Show there exists $k \in K$ such that $\inf_{x \in X} f(x) = f(k)$.

Exercise 3.2 (Extreme value theorem III). Let (X, d) be an open cover compact metric space and $f : X \to \mathbb{R}$ be a continuous function. Show $-\infty < \inf f \leq \sup f < \infty$ and there exists $a, b \in X$ such that $f(a) = \inf f$ and $f(b) = \sup f$. **Hint:** use Exercise 2.10 and Theorem 9.47.

Exercise 3.3 (Dini's Theorem). Let (X, d) be a open cover compact metric space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x \in X} f_n(x) \downarrow 0$ as $n \to \infty$. **Hint:** Given $\varepsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \varepsilon\}$.

Exercise 3.4. Prove Proposition 9.62, i.e. if X is path connected then X is connected.. **Hint:** for sake of contradiction suppose that A and B are two non-empty separated subsets of X such that $X = A \cup B$ and choose a path connecting a point in A to a point in B.

Exercise 3.5. Suppose that $(X, \|\cdot\|)$ is a normed space, $x \in X$, and $R > 0$. Show the open and closed balls in X, $B_x(R)$ and $C_x(R)$, are both convex sets and hence path connected.

Exercise 3.6 (Not to be handed in.). In this exercise we will work inside the metric space $\mathbb{Q}$ with $d(x, y) := |x - y|$ for all $x, y \in \mathbb{Q}$. Let $a, b \in \mathbb{Q}$ with $a < b$ and let

$$J := [a, b] \cap \mathbb{Q} = \{x \in \mathbb{Q} : a \leq x \leq b\}.$$

Show J is disconnected in $\mathbb{Q}$.

4

HomeWork #4 Due Friday, February 1, 2013)

Exercise 4.1 (Existence of Eigenvectors for Symmetric Matrices). Let A be an $n \times n$ symmetric real matrix. Recall this implies (and is equivalent to) $Ax \cdot y = x \cdot Ay$ for all $x, y \in \mathbb{R}^n$ where

$$x \cdot y := \sum_{i=1}^{n} x_i y_i$$

is the usual dot product. In this problem you are going to show that A has an eigenvector by proving the following statements.

1. Show $|Ax \cdot y| \leq C \|x\| \|y\|$ where $C := \sqrt{\sum_{i,j} A_{ij}^2}$ – the Hilbert-Schmidt norm of A.
2. Show $q(x) := Ax \cdot x$ is a continuous function on $\mathbb{R}^n$.
   Let $S := \{x \in \mathbb{R}^n : \|x\| = 1\}$ be the closed and bounded and hence compact unit sphere in $\mathbb{R}^n$. By the extreme value theorem, $\lambda := \max_{x \in S} q(x)$ exists. Let $x_0 \in S$ be chosen so that
   $$\lambda = q(x_0) = Ax_0 \cdot x_0.$$
3. Explain why $R(x) = \frac{Ax \cdot x}{\|x\|^2} = q\left(\frac{x}{\|x\|}\right)$ is a continuous function on $\mathbb{R}^n \setminus \{0\}$ and show x_0 is also a maximizer of R.
4. For any $v \in \mathbb{R}^n$ let $f_v(t) = R(x_0 + tv)$ which is defined for t such that $\|x_0 + tv\| \neq 0$ and in particular for t near zero. Compute $\dot{f}_v(t)$ for t near 0 and in particular show
   $$\dot{f}_v(0) = 2(Ax_0 - \lambda x_0) \cdot v$$
5. Noting that $f_v(t)$ has a maximum at $t = 0$, the first derivative implies $\dot{f}_v(0) = 0$. By making a judicious choice for v, show $Ax_0 = \lambda x_0$, i.e. x_0 is an eigenvector of A with eigenvalue λ.

Exercise 4.2 (Multiplication Operators). Let $V := C([0,1], \mathbb{R})$ and $A : V \to V$ be the operation of multiplication by t, i.e. for $f \in V$ let $Af \in V$ be define by

$$(Af)(t) = tf(t) \text{ for all } t \in [0,1].$$

Show; if there is a $\lambda \in \mathbb{R}$ and $f \in V$ such that $Af = \lambda f$, then $f \equiv 0$, i.e. A has no eigenvectors (or eigenfunctions if you prefer).

[**Moral:** The reader should compare this exercise with Exercise 4.1. Notice that if we define an inner product on V by

$$(f, g) = \int_0^1 f(t) g(t) \, dt$$

then $(Af, g) = (f, Ag)$, i.e. A is symmetric. Nevertheless, A has no eigenvectors. This may be considered as another proof the that unit sphere in V is not compact!]

5

HomeWork #5 Due Friday, February 8, 2013)

Exercise 5.1. Suppose that $f:(a,b)\to\mathbb{R}^n$ and $g:(a,b)\to\mathbb{R}^n$ are differentiable at $t_0\in(a,b)$, show

$$\frac{d}{dt}(f\cdot g)(t_0)=f'(t_0)\cdot g(t_0)+f(t_0)\cdot g'(t_0)$$

where now "$\cdot$" denotes the usual dot product in $\mathbb{R}^n$, i.e.

$$x\cdot y:=\sum_{i=1}^n x_iy_i \text{ for all } x,y\in\mathbb{R}^n. \tag{5.1}$$

Exercise 5.2. Let $f:(a,b)\to\mathbb{R}^n$ and write $f(t)=(f_1(t),\dots,f_n(t))$ where $f_k:(a,b)\to\mathbb{R}$ are functions for $1\le k\le n$. Show; if $\dot f(t)$ exists at $t\in(a,b)$, then $\dot f_k(t)$ exists for all k and

$$\dot f(t)=\left(\dot f_1(t),\dots,\dot f_n(t)\right).$$

Hint: $f_k(t)=e_k\cdot f(t)$ where e_k is the kth standard basis vector for $\mathbb{R}^n$

Exercise 5.3. Suppose that $f:(a,b)\to\mathbb{R}^n$ and write $f(t)=(f_1(t),\dots,f_n(t))$ where $f_k:(a,b)\to\mathbb{R}$. Show; if $\dot f_k(t)$ exists at $t\in(a,b)$ for all k, then $\dot f(t)$ exists and

$$\dot f(t)=\left(\dot f_1(t),\dots,\dot f_n(t)\right).$$

Hint: $f(t)=\sum_{k=1}^n f_k(t)\,e_k$.

Exercise 5.4. For $z\in\mathbb{C}$ recall we have defined

$$e^z:=\sum_{n=0}^\infty\frac{z^n}{n!}.$$

The goal here is to give another proof of Proposition 7.23 using the following outline. Show;

1. $\frac{d}{dt}e^{tz}=e^{tz}$ with $e^{0z}=1$.
2. Use this results and the obvious product rule to show

$$\frac{d}{dt}\left[e^{tz}e^{tw}e^{-t(z+w)}\right]=0 \text{ for all } z,w\in\mathbb{C} \text{ and } t\in\mathbb{R}. \tag{5.2}$$

3. Use Eq. (10.17) to show $e^ze^we^{-(z+w)}=1$ for all $z,w\in\mathbb{C}$.
4. Now, using item 3., show $e^{-z}=(e^z)^{-1}$ and then that $e^ze^w=e^{z+w}$ for all $z,w\in\mathbb{C}$.

Exercise 5.5. Show;

1. $\cos^2y+\sin^2y=1$ for all $y\in\mathbb{R}$. **Hint:** compute $\frac{d}{dy}\left(\cos^2y+\sin^2y\right)=0$.

2. Conclude that $\left|e^{iy}\right| = 1$ for all $y \in \mathbb{R}$.
3. Show $|e^z| = e^{\operatorname{Re} z}$ for all $z \in \mathbb{C}$.

Exercise 5.6 (Theory behind separation of variables). Let $\{a_n\}_{n=-\infty}^{\infty}$ be a summable sequence of complex numbers, i.e. $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. For $t \geq 0$ and $x \in \mathbb{R}$, define

$$F(t,x) = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2} e^{inx} = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2+inx},$$

where as usual $e^{ix} = \cos(x) + i\sin(x)$, this is motivated by replacing x in Eq. (10.18) by ix and comparing the result to Eqs. (10.19) and (10.20).

1. $F(t,x)$ is continuous for $(t,x) \in [0,\infty) \times \mathbb{R}$. **Hint:** Weierstrass M - test, Theorem 7.11.
2. $\partial F(t,x)/\partial t$, $\partial F(t,x)/\partial x$ and $\partial^2 F(t,x)/\partial x^2$ exist for $t > 0$ and $x \in \mathbb{R}$. **Hint:** Use Theorem 10.29 to compute these derivatives. In computing the t derivative, you should let $\varepsilon > 0$ and apply Theorem 10.29 with $t > \varepsilon$ and then afterwards let $\varepsilon \downarrow 0$.
3. F satisfies the heat equation, namely

$$\partial F(t,x)/\partial t = \partial^2 F(t,x)/\partial x^2 \text{ for } t > 0 \text{ and } x \in \mathbb{R}.$$

Exercise 5.7. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that f is differentiable to all orders on $\mathbb{R} \setminus \{0\}$. Further suppose that $\lim_{t\to 0} f^{(n)}(t) = L_n$ exists in $\mathbb{R}$ for each $n \in \mathbb{N}$. Show f is differentiable at $t = 0$ to all orders and that $f^{(n)}(0) = L_n$. Hint: use the mean value theorem.

6

HomeWork #6 Due Friday, February 15, 2013)

Exercise 6.1. Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by

$$f(t):=\begin{cases}0 & \text{if } t\le 0\\ e^{-1/t} & \text{if } t>0\end{cases}.$$

Show f is differentiable to all orders on $\mathbb{R}$ and that $f^{(n)}(0)=0$ for all $n\in\mathbb{N}$.

Hints: first show there exists a polynomial functions, $p_n(x)$, such that $f^{(n)}(t)=p_n\left(t^{-1}\right)f(t)$ for $t>0$. You may also use $e^x\ge\sum_{n=0}^{N}\frac{x^n}{n!}$ when $x\ge 0$ and therefore $\lim_{x\to\infty}\frac{x^n}{e^x}=0$ for all $n\in\mathbb{N}$.

Exercise 6.2 (Taylor's Theorem II). Suppose that $f:\mathbb{R}\to\mathbb{R}^d$ is $n+1$ times differentiable on (a,b) and $\alpha\in(a,b)$. Let

$$p_n(t):=\sum_{k=0}^{n}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

as in Eq. (10.23). Then for all $\beta\in(a,b)$ there exists a c between α and β such that

$$\|f(\beta)-p_n(\beta)\|\le\left\|f^{(n+1)}(c)\right\|\frac{|\beta-\alpha|^{n+1}}{(n+1)!}.$$

Hint: use the idea in the proof of Theorem 10.19 to reduce this to the case where $d=1$.

Exercise 6.3. Prove Corollary 10.44, i.e. for $a>0$ and $b\in\mathbb{R}$,

$$\frac{d}{dx}a^x=\ln a\cdot a^x \text{ for all } x\in\mathbb{R}$$

and

$$\frac{d}{dx}x^b=bx^{b-1} \text{ for all } x>0.$$

Exercise 6.4. Use Taylor's Theorem 10.38 to show

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n \text{ for } -\frac{1}{2}<x<1.$$

Exercise 6.5. Show the series

$$f(x):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$$

converges for $|x|<1$ and that $f(x)=\ln(1+x)$ for $|x|<1$. **Hint:** this problem can be done independently from Exercise 6.4. Your goal is to show $g(x):=f(x)-\ln(1+x)=0$ for $|x|<1$.

Exercise 6.6 (Binomial Expansions). Let $a \in \mathbb{R}$ and $f(x) = (1+x)^a$. By direct computation using induction one shows, $f(0) = 1$, $f'(x) = a(1+x)^{a-1}$,

$$\begin{aligned}
\frac{f''(x)}{2!} &= \frac{a(a-1)}{2}(1+x)^{a-2}, \\
\frac{f^{(3)}(x)}{3!} &= \frac{a(a-1)(a-2)}{3!}(1+x)^{a-2} \\
&\vdots \\
\frac{f^{(n)}(x)}{n!} &= \frac{a(a-1)\dots(a-n+1)}{n!}(1+x)^{a-n} \\
&= \binom{a}{n}(1+x)^{a-n}.
\end{aligned}$$

So we expect, based on Taylor's Theorem 10.38, that

$$(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n. \tag{6.1}$$

for $|x| < R$ where R is the radius of convergence of the series above. Show;

1. $R = 1$. Hint: you might use the ratio test here.
2. Show $(1+x)^a$ solves the differential equation,

$$(1+x)\frac{d}{dx}(1+x)^a = a(1+x)^a \text{ for } x > -1. \tag{6.2}$$

6.1

3. Show Eq. (~~10.29~~) is valid for $|x| < 1$ as follows.
   a) Show

$$g(x) := \sum_{n=0}^{\infty} \binom{a}{n} x^n \tag{6.3}$$

   satisfies $(1+x)g'(x) = ag(x)$ for $|x| < 1$, the same differential equation as $(1+x)^a$.
   b) Next compute the derivative of $h(x) := (1+x)^{-a} g(x)$ and use to show $h(x) = 1$ for $|x| < 1$.