C

Appendix: Math 140A Topics

C.1 Summary of Key Facts about Real Numbers

1. The real numbers, $\mathbb{R}$, is the unique (up to order preserving field isomorphism) ordered field with the least upper bound property or equivalently which is Cauchy complete.
2. Informally the real numbers are the rational numbers with the (irrational) hole filled in.
3. Monotone bounded sequence always converge in $\mathbb{R}$.
4. A sequence converges in $\mathbb{R}$ iff it is Cauchy.
5. Cauchy sequences are bounded.
6. $\mathbb{N}$ is unbounded from above in $\mathbb{R}$.
7. For all $\varepsilon > 0$ in $\mathbb{R}$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.
8. $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. In particular, between any two real numbers $a < b$, there are infinitely many rational and irrational numbers.
9. Decimal numbers map (almost 1-1) into the real numbers by taking the limit of the truncated decimal number.
10. If $a, b, \varepsilon \in \mathbb{R}$, then
    a) $a \le b$ by showing that $a \le b + \varepsilon$ for all $\varepsilon > 0$.
    b) $a = b$ by proving $a \le b$ and $b \le a$ or
    c) $a = b$ by showing $|b - a| \le \varepsilon$ for all $\varepsilon > 0$.
11. A number of standard limit theorems hold, see Theorem 3.13.
12. Unlike limits, $\limsup$ and $\liminf$ always exist. Moreover we have; $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$ with equality iff $\lim_{n\to\infty} a_n$ exists in which case
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$
We may allow the values of $\pm\infty$ in these statements.
13. If $b_k := \{a_{n_k}\}_{k=1}^{\infty}$ is a convergent subsequence of $\{a_n\}$, then
$$\liminf_{n\to\infty} a_n \le \lim_{k\to\infty} b_k \le \limsup_{n\to\infty} a_n$$
and we may choose $\{b_k\}$ so that $\lim_{k\to\infty} b_k = \limsup_{n\to\infty} a_n$ or $\lim_{k\to\infty} b_k = \liminf_{n\to\infty} a_n$.
14. Bounded sequences of real numbers always have convergence subsequences.
15. If $S \subset \mathbb{R}$ and $A := \sup(S)$, then there exists $\{a_n\}_{n=1}^{\infty} \subset S$ such that $a_n \le a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = \sup(S)$.
16. If $S \subset \mathbb{R}$ and $A := \inf(S)$, then there exists $\{a_n\}_{n=1}^{\infty} \subset S$ such that $a_{n+1} \le a_n$ for all n and $\lim_{n\to\infty} a_n = \inf(S)$.

C.2 Test 2 Review Topics:

1. Understand the basic properties of **complex numbers**.
2. **Coutability**. Key facts are that countable union of countable sets is countable and the **finite** product of countable sets is countable.
3. Definitions of **metric** and **normed spaces** and their basic properties which in the end of the day typically follow from the **triangle inequality**.
4. You should know that metrics and norms are continuous functions that satisfy,
$$|\|x\| - \|y\|| \le \|x - y\| \text{ and}$$
$$|d(x,y) - d(x',y')| \le d(x,x') + d(y,y').$$
5. Be aware of different norms, $\|\cdot\|_u$, $\|\cdot\|_1$, and $\|\cdot\|_2$.
6. Understand the notion of; **limits** of sequences, **Cauchy** sequences, **completeness**, limits and **continuity** of functions.
7. Know what is meant by **pointwise** and **uniform convergence**. You should be able to compute pointwise limits and know how to test if the limit is uniform or not. A key theorem is the uniform limit of continuous functions is still continuous.

C.3 After Test 2 Review Topics:

Let $(X, \|\cdot\|)$ be a Banach space.

1. Know $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$ if the limit exists.
2. Know $\sum_{n=1}^{\infty} x_n$ converges absolutely iff $\sum_{n=1}^{\infty} \|x_n\| < \infty$ and absolute convergence implies convergence in a Banach space.
3. Telescoping series and geometric series.
4. Alternating series test.

5. Absolute convergence tests: 1) integral test, 2) root test, 3) ratio test often combined with the 4) comparison test.
6. $p-$ series examples.
7. $n^{th}-$ term test for divergence.
8. Cauchy criteria for convergence and the fact that tails of convergent series tend to zero. i.e. tails of convergent series tend to zero.
9. Pointwise convergence of sums.
10. Uniform convergence of sums and the Weierstrass $M-$ test
11. Power series including radius of convergence notion.
12. The exponential function and its relatives, sin, sinh, cos, cosh.