7

Series and Sums in Banach Spaces

Definition 7.1. *Suppose $(X, \|\cdot\|)$ is a normed space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in X. Then we say $\sum_{n=1}^{\infty} x_n$* ***converges*** *in X iff $s := \lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists in X otherwise we say $\sum_{n=1}^{\infty} x_n$ diverges. We often let $S_N := \sum_{n=1}^{N} x_n$ and refer to $\{S_N\}_{N=1}^{\infty} \subset X$ as the sequence of partial sums.*

If $X = \mathbb{R}$ and $x_n \geq 0$, then $\sum_{n=1}^{\infty} x_n$ diverges iff $\lim_{N\to\infty} \sum_{n=1}^{N} x_n = \infty$ and so we will write $\sum_{n=1}^{\infty} x_n = \infty$ in this case to indicate that $\sum_{n=1}^{\infty} x_n$ diverges to infinity.

Theorem 7.2 (Comparison Theorem). *Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences in $[0, \infty)$. If $a_n \leq b_n$ for all n, then*

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

where we allow for these sums to be infinite. Moreover if $a_n \leq b_n$ for a.a. n then $\sum_{n=1}^{\infty} b_n < \infty$ implies $\sum_{n=1}^{\infty} a_n < \infty$ and if $\sum_{n=1}^{\infty} a_n = \infty$ then $\sum_{n=1}^{\infty} b_n = \infty$.

Proof. Let $A_k := \sum_{n=1}^{k} a_n$ and $B_k := \sum_{n=1}^{k} b_n$. Then a simple induction argument shows that $A_k \leq B_k$ for all k and therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} A_k \leq \lim_{k\to\infty} B_k = \sum_{n=1}^{\infty} b_n$$

by the sandwich lemma. ■

Theorem 7.3 (Telescoping Series / Fundamental Theorem of Summation). *Let $\{f(n)\}_{n=1}^{\infty} \subset X$ be a sequence, then*

$$\sum_{n=1}^{N} [f(n+1) - f(n)] = f(N+1) - f(1) \text{ for all } N \in \mathbb{N} \tag{7.1}$$

and $\sum_{n=1}^{\infty} [f(n+1) - f(n)]$ is convergent in X iff $\lim_{N\to\infty} f(N)$ exists in X in which case,

$$\sum_{n=1}^{\infty} (f(n+1) - f(n)) = \lim_{N\to\infty} f(N) - f(1).$$

We also have,

$$\sum_{n=M}^{N} [f(n+1) - f(n)] = f(N+1) - f(M) \text{ for } N \geq M. \tag{7.2}$$

Proof. When $N = 3$ we have,

$$\begin{aligned}\sum_{n=1}^{3} (f(n+1) - f(n)) &= (f(2) - f(1)) + (f(3) - f(2)) + (f(4) - f(3)) \\ &= f(4) - f(1).\end{aligned}$$

In general, Eqs. (7.1) and (7.2) are easily verified by a simple induction argument. The rest of the theorem is now evident. ■

Example 7.4 (Geometric Series). Suppose that $f(n) = \alpha^n$ where $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}f(n+1) - f(n) &= \alpha^{n+1} - \alpha^n \\ &= \alpha^n (\alpha - 1)\end{aligned}$$

and we find,

$$(\alpha - 1) \sum_{n=1}^{N} \alpha^n = \alpha^{N+1} - \alpha.$$

If $\alpha \neq 1$ it follows that

$$\sum_{n=1}^{N} \alpha^n = \frac{\alpha^{N+1} - \alpha}{\alpha - 1}$$

and if $|\alpha| < 1$, it follows that $\left|\alpha^{N+1}\right| = |\alpha|^{N+1} \to 0$ as $N \to \infty$ and therefore

$$\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1 - \alpha}.$$

Theorem 7.5 (Integral test). *Let $f : (0, \infty) \to \mathbb{R}$ be a C^1 function such that $f'(x) \geq 0$ and $f'(x)$ is decreasing in x (i.e. $f''(x) \leq 0$ if it exists) and $f(\infty) := \lim_{x\to\infty} f(x)$, see Figure 7.1. (As f is an increasing function, $f(\infty)$ exists in $(-\infty, \infty]$.) Then*

$$[f(N+1)-f(M)] \le \sum_{n=M}^{N} f'(n) \le [f(N+1)-f(M)] + f'(M) - f'(N+1) \tag{7.3}$$
$$\le [f(N+1)-f(M)] + f'(M)$$

for all $M \le N$. Letting $N \to \infty$ in these inequalities also gives,

$$f(\infty) - f(M) \le \sum_{n=M}^{\infty} f'(n) \le f(\infty) - f(M) + f'(M) \tag{7.4}$$

and in particular, $\sum_{n=1}^{\infty} f'(n) < \infty$ iff $f(\infty) < \infty$.

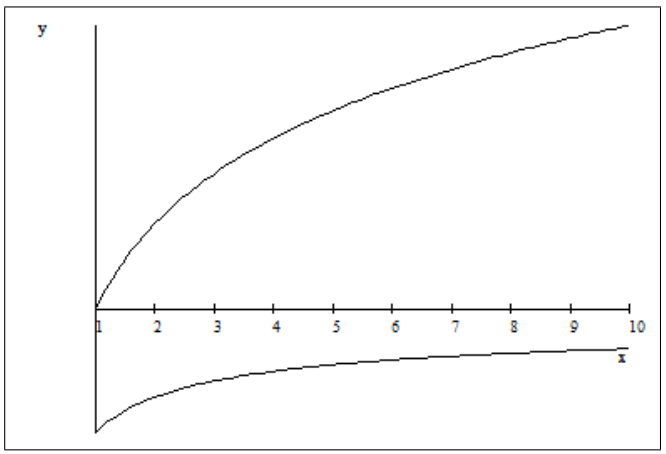

Fig. 7.1. The items plotted here are $\ln x$ and $-1/\sqrt{x}$ both of which satisfy the hypothesis of the propostion.

Proof. By the mean value theorem, there exists $c_n \in (n-1, n)$ such that $f(n) - f(n-1) = f'(c_n)$. Since f' is decreasing, it follows that

$$f'(n) \le f'(c_n) \le f'(n-1) \text{ for all } n,$$

or equivalently[1] that

$$f'(n) \le f(n) - f(n-1) \le f'(n-1) \text{ for all } n.$$

[1] This is a standard inequality involving convex functions. The reader should draw the picture and note that $f(n) - f(n-1)$ is the slope of the cord joining $(n-1, f(n-1))$ to $(n, f(n))$.

Summing these inequalities on n, using

$$\sum_{n=M+1}^{N+1} [f(n) - f(n-1)] = f(N+1) - f(M),$$

shows,

$$\sum_{n=M+1}^{N+1} f'(n) \le f(N+1) - f(M) \le \sum_{n=M+1}^{N+1} f'(n-1) = \sum_{n=M}^{N} f'(n).$$

This inequality is equivalent to Eq. (7.3) because it gives,

$$[f(N+1)-f(M)] \le \sum_{n=M}^{N} f'(n) = \sum_{n=M+1}^{N+1} f'(n) + f'(M) - f'(N+1)$$
$$\le [f(N+1)-f(M)] + f'(M) - f'(N+1).$$

■

Corollary 7.6 (p − series). *Let $p \in \mathbb{R}$, then*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \infty & \text{if } p \le 1 \\ < \infty & \text{if } p > 1 \end{cases}.$$

Proof. If $p \le 0$, then $\lim_{n\to\infty} \frac{1}{n^p} \ne 0$ and hence the series diverges and we may assume that $p > 0$. For $p > 1$, let $f(x) = -x^{-(p-1)}$ so that $f'(x) = (p-1)x^{-p}$ which is decreasing in x and therefore by Theorem 7.5,

$$f(\infty) - f(M) \le \sum_{n=M}^{\infty} \frac{p-1}{n^p} \le f(\infty) - f(M) + f'(M) = \frac{1}{M^{p-1}} + \frac{p-1}{M^p} < \infty$$

and so

$$\frac{1}{(p-1)M^{p-1}} \le \sum_{n=M}^{\infty} \frac{1}{n^p} \le \frac{1}{(p-1)M^{p-1}} + \frac{1}{M^p}.$$

In particular for $M = 1$,

$$\frac{1}{p-1} \le \sum_{n=1}^{\infty} \frac{1}{n^p} \le \frac{1}{p-1} + 1.$$

Since

$$\frac{1}{p-1} \le \sum_{n=1}^{\infty} \frac{1}{n^p} \le \sum_{n=1}^{\infty} \frac{1}{n}$$

we may let $p \downarrow 1$ in order to show, $\infty \le \sum_{n=1}^{\infty} \frac{1}{n}$. This complete the proof as for any $p < 1$ we have ,

$$\infty \le \sum_{n=1}^{\infty} \frac{1}{n} \le \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

■

Exercise 7.1. Take $f(x) = \ln x$ in Theorem 7.5 in order to directly conclude that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Exercise 7.2. Let $0 \le p < \infty$. Prove,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} = \begin{cases} \infty & \text{if } 0 \le p \le 1 \\ < \infty & \text{if } \quad p > 1 \end{cases}.$$

by applying Theorem 7.5 with the following functions;

1. for $0 \le p < 1$ take $f(x) = (\ln x)^{1-p}$,
2. for $p = 1$ take $f(x) = \ln \ln x$, and
3. for $p > 1$ take $f(x) = -(\ln x)^{1-p}$.

Theorem 7.7. *Let $(X, \|\cdot\|)$ be a Banach space and $\{x_k\}_{k=1}^{\infty} \subset X$ be a sequence. Then;*

1. $\sum_{k=1}^{\infty} x_k$ *converges iff*

$$\left\| \sum_{k=m}^{n} x_k \right\| \to 0 \text{ as } n, m \to \infty \text{ with } n \ge m.$$

2. *If* $\sum_{k=1}^{\infty} x_k$ *converges then* $\lim_{k\to\infty} x_k = 0$ *or alternatively if* $\lim_{k\to\infty} x_k \ne 0$ *then* $\sum_{k=1}^{\infty} x_k$ *diverges.*
3. *If* $\sum_{k=1}^{\infty} x_k$ *converges then* $\lim_{N\to\infty} \sum_{k=N}^{\infty} x_k = 0$, *i.e. the N - tail,* $\sum_{k=N}^{\infty} x_k$, *of a convergent series,* $\sum_{k=1}^{\infty} x_k$, *go to zero as* $N \to \infty$.

Proof. Let $S_n := \sum_{k=1}^{n} x_k$ so that $\sum_{k=1}^{\infty} x_k$ converges iff $\lim_{n\to\infty} S_n$ exists iff $\{S_n\}_{n=1}^{\infty}$ is Cauchy since X is a Banach space which gives item 1. since $S_n - S_{m-1} = \sum_{k=m}^{n} x_k$. For the second item apply the first with $n = m+1$. For the third item let $S := \sum_{k=1}^{\infty} x_k$, then $\lim_{N\to\infty} S_N = S$ and so by very definition,

$$\sum_{k=N}^{\infty} x_k = S - S_{N+1} \to 0 \text{ as } N \to \infty.$$

■

Exercise 7.3. Let (X, d) be a metric space. Suppose that $\{x_n\}_{n=1}^{\infty} \subset X$ is a sequence and set $\varepsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \le \sum_{k=n}^{m-1} \varepsilon_k \le \sum_{k=n}^{\infty} \varepsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^{\infty} \varepsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$ then

$$d(x, x_n) \le \sum_{k=n}^{\infty} \varepsilon_k.$$

Theorem 7.8 (Absolute Convergence Implies Convergence). *Let $(X, \|\cdot\|)$ be a Banach space $\{x_k\}_{k=1}^{\infty} \subset X$ be a sequence. Then $\sum_{k=1}^{\infty} \|x_k\| < \infty$ then $\sum_{k=1}^{\infty} x_k$ is convergent. [We say $\sum_{k=1}^{\infty} x_k$ is* ***absolutely convergent*** *if $\sum_{k=1}^{\infty} \|x_k\| < \infty$.]*

Exercise 7.4. Prove Theorem 7.8. Namely if $(X, \|\cdot\|)$ is a Banach space and $\{x_k\}_{k=1}^{\infty} \subset X$ is a sequence, then $\sum_{k=1}^{\infty} \|x_k\| < \infty$ implies $\sum_{k=1}^{\infty} x_k$ is convergent.

Proposition 7.9 (Alternating Series Test). *If $\{a_k\}_{k=1}^{\infty} \subset [0, \infty)$ is a non-increasing sequence (i.e. $a_k \ge a_{k+1}$ for all k) such that $\lim_{k\to\infty} a_k = 0$, then $s := \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is convergent. Moreover, for all $n \in \mathbb{N}$*

$$\left| s - \sum_{k=1}^{n} (-1)^{k+1} a_k \right| = \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k \right| \le a_{n+1}. \tag{7.5}$$

Proof. The proof is essentially contained in Figure 7.2
Let $S_n := \sum_{k=1}^{n} (-1)^{k+1} a_k$. For $n \in \mathbb{N}$,

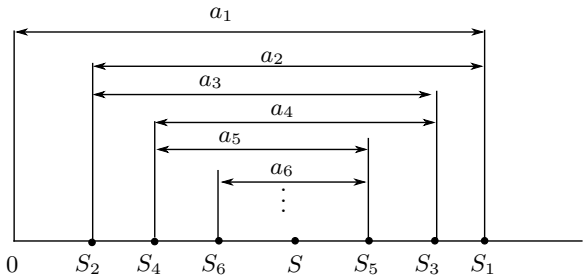

Fig. 7.2. The picture proof of the alternating series test.

$$S_{2n+1} = S_{2n-1} - (a_{2n} - a_{2n+1}) \le S_{2n-1} \text{ and}$$
$$S_{2(n+1)} = S_{2n} + (a_{2n+1} - a_{2n+2}) \ge S_{2n},$$

wherein we have used the fact that $\{a_k\}$ is decreasing to conclude the terms in parenthesis above are non-negative. From these inequalities we learn that $\{S_{2n}\}_{n=1}^{\infty}$ is an increasing sequence while $\{S_{2n+1}\}_{n=1}^{\infty}$ is a decreasing sequence and hence

$$s_{\text{ev}} := \lim_{n\to\infty} S_{2n} \text{ exists in } (-\infty,\infty] \text{ and}$$
$$s_{\text{odd}} := \lim_{n\to\infty} S_{2n-1} \text{ exists in } (-\infty,\infty].$$

Since $S_{2n+1} - S_{2n} = a_{2n+1} \to 0$ as $n \to \infty$ it follows that $s_{\text{ev}} = s_{\text{odd}} \in \mathbb{R}$. It is now easy to conclude (You prove!) that

$$S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k = \lim_{n\to\infty} S_n = s_+ = s_- \in \mathbb{R}.$$

In summary, $S_{2n} \uparrow S$ while $S_{2n+1} \downarrow S$ as $n \to \infty$ and in particular, for all $n \in \mathbb{N}$,

$$S_{2n+1} - a_{2n} = S_{2n} \le S \le S_{2n+1} = S_{2n} + a_{2n+1}.$$

These inequalities then implies

$$0 \le S - S_{2n} = \sum_{k=2n+1}^{\infty} (-1)^{k+1} a_k \le a_{2n+1}$$

and

$$0 \le S_{2n+1} - S = \sum_{k=2n}^{\infty} (-1)^{k+1} a_k \le a_{2n}.$$

which imply Eq. (7.5). **Alternatively,**

$$0 \le a_1 - a_2 = S_2 \le S \le S_1 \le a_1$$

so that $0 \le S \le a_1$. Applying these results to $\sum_{k=n+1}^{\infty} (-1)^{k+1} a_k$ shows,

$$\left| s - \sum_{k=1}^{n} (-1)^{k+1} a_k \right| = \left| \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k \right| \le a_{n+1}.$$

■

Example 7.10. By the alternating series test,

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^p}$$

is convergent for all $p > 0$. On the other hand, by Corollary 7.6, the series is absolutely convergent iff $p > 1$.

Theorem 7.11 (Uniform convergence and the Weierstrass $M-$ test). *Suppose that $(X, \|\cdot\|)$ is a Banach space, (Y, d) is a metric space, for each $n \in \mathbb{N}$, $f_n : Y \to X$ is a function, and there exists $\{M_n\}_{n=1}^{\infty} \subset [0,\infty)$ satisfying;*

$$\sup_{y\in Y} \|f_n(y)\|_X \le M_n \ \forall\ n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} M_n < \infty.$$

Then $S_N(y) := \sum_{n=1}^{N} f_n(y)$ converges absolutely and uniformly to $S(y) = \sum_{n=1}^{\infty} f_n(y)$. Moreover, if we further assume that $\{f_n\}_{n=1}^{\infty} \subset C(Y,X)$ (i.e. f_n is continuous for all n), then the function $S : Y \to X$ is also continuous.

Proof. For any $y \in Y$,

$$\sum_{n=1}^{\infty} \|f_n(y)\|_X \le \sum_{n=1}^{\infty} M_n < \infty$$

and therefore $S(y) = \sum_{n=1}^{\infty} f_n(y)$ converges absolutely. Moreover we have,

$$\|S(y) - S_N(y)\| = \lim_{M\to\infty} \|S_M(y) - S_N(y)\| = \lim_{M\to\infty} \left\| \sum_{n=N+1}^{M} f_n(y) \right\|$$
$$\le \liminf_{M\to\infty} \sum_{n=N+1}^{M} \|f_n(y)\| = \sum_{n=N+1}^{\infty} \|f_n(y)\| \le \sum_{n=N+1}^{\infty} M_n.$$

As the last member of this inequality does not depend on y we have,

$$\sup_{y\in Y}\|S(y)-S_N(y)\|\le\sum_{n=N+1}^{\infty}M_n\to 0 \text{ as } N\to\infty$$

because tails of convergent series vanish, Theorem 7.7. The continuity of S now follows form the continuity of S_N, the uniform convergence just proved, and Theorem 6.48. ■

Theorem 7.12 (Root test). *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{C}$ and let $\alpha:=\limsup_{n\to\infty}|a_n|^{1/n}$. Then*

1. *If $\alpha<1$ then $\sum_{n=1}^{\infty}|a_n|<\infty$ and $\sum_{n=1}^{\infty}a_n$ is absolutely convergent.*
2. *If $\alpha>1$, then $\limsup_{n\to\infty}|a_n|=\infty$ and $\sum_{n=1}^{\infty}a_n$ diverges.*
3. *If $\alpha=1$, the test **fails**, i.e. you must work harder!]*

Proof. We take each item in turn.

1. If $\alpha<1$, let $\beta\in(\alpha,1)$, then $|a_n|^{1/n}\le\beta$ for a.a. n which implies that $|a_n|\le\beta^n$ for a.a. n and so the result follows by the comparison Theorem 7.2 as
$$\sum_{n=1}^{\infty}\beta^n=\frac{\beta}{1-\beta}<\infty.$$
2. If $\alpha>1$ and $\beta\in(1,\alpha)$, then $|a_n|^{1/n}\ge\beta$ i.o. n and hence $|a_n|\ge\beta^n$ i.o. n. As $\beta^n\to\infty$ as $n\to\infty$ it follows that $\limsup_{n\to\infty}|a_n|=\infty$.
3. From Corollary 7.6 we know that $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ while $\sum_{n=1}^{\infty}\frac{1}{n^2}<\infty$. However from Lemma 3.30,
$$\lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/n}=1=\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{1/n}$$
which shows the test has failed. In fact, if $0<p<\infty$ and $k\in\mathbb{N}$ such that $k\ge p$, then $\frac{1}{n^k}\le\frac{1}{n^p}\le 1$ which implies
$$\left(\frac{1}{n^k}\right)^{1/n}\le\left(\frac{1}{n^p}\right)^{1/n}\le(1)^{1/n}=1.$$
By Lemma 3.30 we know $\lim_{n\to\infty}\left(\frac{1}{n^k}\right)^{1/n}=1$ and therefore by the sandwich lemma, $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)^{1/n}=1$.

■

Exercise 7.5. For every $p\in\mathbb{N}$, show $\sum_{n=0}^{\infty}(n)^{n/p}z^n$ is convergent iff $z=0$.

In what follows let $(Z,\|\cdot\|)$ be a complex Banach space. For example, $Z=\mathbb{C}$ and $\|z\|=|z|$ is an important special case.

Theorem 7.13 (Ratio test). *Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{C}$ such that $a_n\ne 0$ for a.a. n. Then*

1. *If $\alpha:=\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ then $\sum_{n=1}^{\infty}|a_n|<\infty$ and $\sum_{n=1}^{\infty}a_n$ is absolutely convergent.*
2. *If $\left|\frac{a_{n+1}}{a_n}\right|\ge 1$ for a.a. n then $\lim_{n\to\infty}|a_n|>0$ and $\sum_{n=1}^{\infty}a_n$ diverges.*
3. *If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ then $\lim_{n\to\infty}|a_n|=\infty$ and $\sum_{n=1}^{\infty}a_n$ diverges.*
4. *If $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, the test **fails**, i.e. you must work harder!]*

Proof. We take each item in turn.

1. If $\alpha<1$, let $\beta\in(\alpha,1)$, then $\left|\frac{a_{n+1}}{a_n}\right|\le\beta$ for a.a. n, i.e. there exists $N\in\mathbb{N}$ such that $|a_{n+1}|\le\beta|a_n|$ for all $n\ge N$. A simple induction argument shows,
$$|a_n|\le|a_N|\beta^{n-N}=\beta^{-N}|a_N|\beta^n \text{ for } n\ge N.$$
The result follows by the comparison Theorem 7.2 and the fact that
$$\sum_{n=N}^{\infty}\beta^{-N}|a_N|\beta^n=\frac{|a_N|}{1-\beta}<\infty.$$
2. Suppose there exists $N\in\mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right|\ge 1$ for all $n\ge N$. This inequality says that $|a_n|$ is non-decreasing for large n and therefore $\lim_{n\to\infty}|a_n|\ge|a_m|$ for any $m\ge N$. We may now choose $m\ge N$ such that $|a_m|\ne 0$.
3. If $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>\beta>1$, then $\left|\frac{a_{n+1}}{a_n}\right|\ge\beta$ for a.a. n. Working as above there exists $N\in\mathbb{N}$ such that $|a_N|\ne 0$ and $|a_n|\ge|a_N|\beta^{n-N}$ for all $n\ge N$. From this it follows that $\lim_{n\to\infty}|a_n|=\infty$.
4. From Corollary 7.6 we know that $\sum_{n=1}^{\infty}\frac{1}{n^p}<\infty$ iff $p>1$. However
$$\lim_{n\to\infty}\left(\frac{\left(\frac{1}{n+1}\right)^p}{\left(\frac{1}{n}\right)^p}\right)=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^p=\left[\lim_{n\to\infty}\left(\frac{n}{n+1}\right)\right]^p=1^p=1$$
which shows the test has failed and does not resolve when $\sum_{n=1}^{\infty}\frac{1}{n^p}<\infty$.

■

Definition 7.14. *Given $z_0\in\mathbb{C}$ and $\{a_n\}_{n=0}^{\infty}\subset Z$, the series of the form*
$$\sum_{n=0}^{\infty}a_n(z-z_0)^n \tag{7.6}$$

is called a ***power series****. If $z_0 = 0$ we call it a* ***Maclaurin series****, i.e. a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n. \tag{7.7}$$

The ***radius of convergence*** *of either of these series is defined to be $R := \frac{1}{\alpha} \in [0, \infty]$ where*

$$\alpha := \limsup_{n\to\infty} \|a_n\|^{1/n} \in [0, \infty].$$

By definition, $\frac{1}{0} := \infty$ in this formula.

The next theorem shows that R is the critical radius governing the convergence of Eqs. (7.6) and (7.7).

Proposition 7.15. *If R is the radius of convergence of a power series in Eq. (7.6) then:*

1. *If $|z - z_0| < R$, the series converges.*
2. *If $|z - z_0| > R$, the series diverges.*
3. *If $|z - z_0| = R$, the series may or may not converge.*

We may then also characterize R as

$$R := \sup\left\{|z - z_0| : z \in \mathbb{C} \text{ and } \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ converges}\right\}. \tag{7.8}$$

Proof. Let

$$\begin{aligned}\rho(z) &:= \limsup_{n\to\infty} \|a_n (z - z_0)^n\|^{1/n} \\ &= |z - z_0| \cdot \limsup_{n\to\infty} \|a\|^{1/n} = \frac{|z - z_0|}{R}.\end{aligned}$$

By the root test, Theorem 7.12, we know that power series in Eq. (7.6) converges if $\rho(z) < 1$ and diverges if $\rho(z) > 1$ and the test fails if $\rho(z) = 1$. ■

Corollary 7.16. *If $\mu = \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|}$ exists, the radius (R) of convergence of a power series in Eq. (7.6) is $R = \frac{1}{\mu} = \lim_{n\to\infty} \frac{\|a_n\|}{\|a_{n+1}\|}$.*

Proof. If $\mu = \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|}$, then

$$\rho(z) := \lim_{n\to\infty} \frac{\left\|a_{n+1}(z - z_0)^{n+1}\right\|}{\|a_n (z - z_0)^n\|} = |z - z_0| \lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} = \mu \cdot |z - z_0|.$$

The the Ratio test in Theorem 7.13 now implies that on test implies that $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges if $|z - z_0| < \frac{1}{\mu}$ $(\rho(z) < 1)$ and diverges if $|z - z_0| > \frac{1}{\mu}$ $(\rho(z) > 1)$, i.e. $R = \frac{1}{\mu}$ from Eq. (7.8). ■

Theorem 7.17. *If the radius of convergence (R) of a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is positive $(R > 0)$, then the functions*

$$S : D(z_0, R) := \{z \in \mathbb{C} : |z - z_0 < R|\} \to Z$$

defined by $S(z) := \sum_{n=0}^{\infty} a_n (z - z_0)^n$ is continuous and the series is uniformly convergent on $D(z_0, \rho)$ for all $\rho < R$.

Proof. For any $\rho < R$ let $M_n := \|a_n\| \rho^n$. Then $\sum_{n=0}^{\infty} M_n < \infty$ by the root test or the definition of R. So if $z \in D(z_0, \rho)$ we find,

$$\|a_n (z - z_0)^n\| \le \|a_n\| \rho^n = M_n.$$

It follows from the Weierstrass M – test, Theorem 7.11, that the series is uniformly convergent and S is continuous on $D(z_0, \rho)$ for all $\rho < R$. Since $D(z_0, R) = \cup_{0<\rho<R} D(z_0, \rho)$, we may conclude that S is continuous on $D(z_0, \rho)$. ■

Exercise 7.6. Show that each of the following power series have an infinite radius of convergence and hence define continuous functions from $\mathbb{C}$ to $\mathbb{C}$.

1. $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$,
2. $\sin(z) := \sum_{n=0}^{\infty} (-1)^{2n+1} \frac{z^{2n+1}}{(2n+1)!}$, and
3. $\cos(z) := \sum_{n=0}^{\infty} (-1)^{2n} \frac{z^{2n}}{(2n)!}$.

The reader should verify **Euler's formula**

$$e^{iz} = \cos(z) + i \sin(z).$$

We also define

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \text{ and}$$
$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

The following identities are now easily verified;

$$\begin{aligned}\cos(-z) &= \cos(z) \text{ and } \cosh(-z) = \cosh(z) \\ \sin(-z) &= -\sin(z) \text{ and } \sinh(-z) = -\sinh(z), \\ e^z &= \cosh(z) + \sinh(z), \\ \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{i}\sinh(iz), \text{ and} \\ \cos(z) &= \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz).\end{aligned}$$

Theorem 7.18 (Hilbert Schmidt norm). *Let $Z = \mathbb{C}^{J_N \times J_N}$ denote the space of $N \times N$ complex matrices, $A = (A_{ij})_{i,j=1}^N$ with $A_{ij} \in \mathbb{C}$. We let*

$$\|A\|_2 := \sqrt{\sum_{i,j=1}^N |A_{ij}|^2}$$

which is called the ***Hilbert Schmidt norm*** *on Z. This norm satisfies,*

1. $\|I\|_2 = \sqrt{N}$ *where I is the $N \times N$ – identity matrix.*
2. $\|AB\|_2 \le \|A\|_2 \cdot \|B\|_2$ *for all $A, B \in Z$.*
3. $\|A^n\|_2 \le \|A\|_2^n$ *for all $n \in \mathbb{N}$.*
4. *If $A_n \to A$ and $B_n \to B$ then $A_n B_n \to AB$ and $A_n + B_n \to A + B$ as $n \to \infty$. Thus matrix multiplication and addition are continuous operations on $(Z, \|\cdot\|_1)$.*
5. *If (X, d) is a metric space and $f : X \to Z$ and $g : X \to Z$ are continuous functions then so is $f \cdot g$ (order matters) and $f + g$.*
6. *The functions $f(A) = A^n$ is continuous on Z for all $n \in \mathbb{N}_0$, where by convention $A^0 = I$.*
7. *$(Z, \|\cdot\|)$ is a Banach space.*

Proof. Item 1. is clear. From the definition of matrix multiplication and the Cauchy Schwarz inequality we find

$$\left|(AB)_{ij}\right|^2 = \left|\sum_{k=1}^N A_{ik} B_{kj}\right|^2 \le \sum_{k=1}^N |A_{ik}|^2 \cdot \sum_{k=1}^N |B_{kj}|^2 .$$

Therefore,

$$\begin{aligned}\|AB\|_2^2 &= \sum_{i,j=1}^n \left|(AB)_{ij}\right|^2 \le \sum_{i,j=1}^n \left(\sum_{k=1}^N |A_{ik}|^2 \cdot \sum_{k=1}^N |B_{kj}|^2\right) \\ &= \sum_{i,k=1}^N |A_{ik}|^2 \cdot \sum_{j,k=1}^N |B_{kj}|^2 = \|A\|_2^2 \cdot \|B\|_2^2\end{aligned}$$

which proves item 2. Item 3. follows by an easy induction argument.

Item 4. Let $\delta A_n := A_n - A$ and $\delta B_n := B_n - B$ so that $A_n = A + \delta A_n$ and $B_n = B + \delta B_n$ where $\|\delta A_n\|_2 \to 0$ and $\|\delta B_n\|_2 \to 0$ as $n \to \infty$. Then

$$\begin{aligned}\|A_n B_n - AB\|_2 &= \|(A + \delta A_n)(B + \delta B_n) - AB\|_2 \\ &= \|\delta A_n B + A\delta B_n + \delta A_n \delta B_n\|_2 \\ &\le \|\delta A_n B\|_2 + \|A\delta B_n\|_2 + \|\delta A_n \delta B_n\|_2 \\ &\le \|\delta A_n\|_2 \|B\|_2 + \|A\|_2 \|\delta B_n\|_2 + \|\delta A_n\|_2 \|\delta B_n\|_2 \to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly,

$$\|A_n + B_n - (A + B)\|_2 = \|\delta A_n + \delta B_n\|_2 \to 0 \text{ as } n \to \infty.$$

Item 5. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ and $x_n \to x$ as $n \to \infty$, then using item 4.,

$$\lim_{n\to\infty} (f \cdot g)(x_n) = \lim_{n\to\infty} [f(x_n) g(x_n)] = f(x) g(x) = (f \cdot g)(x)$$

and

$$\lim_{n\to\infty} (f + g)(x_n) = \lim_{n\to\infty} [f(x_n) g(x_n)] = f(x) + g(x) = (f + g)(x).$$

This shows $f \cdot g$ and $f + g$ are continuous.

Item 6. follows from item 5 with $X = Z$ along with an induction argument.

Item 7. follows from the fact that $(Z, \|\cdot\|)$ is $\mathbb{C}^{N^2}$ with the 2 – norm in a slight disguise. ■

Definition 7.19 (Matrix Functions). *If $f(z) = \sum_{n=0}^\infty c_n z^n$ for some $c_n \in \mathbb{C}$ and $A \in \mathbb{C}^{J_N \times J_N}$, then we define*

$$f(A) := \sum_{n=0}^\infty c_n A^n \text{ provided the sum is convergent.}$$

For example,

$$\sin(A) := \sum_{n=0}^\infty \frac{(-1)^{2n+1}}{n!} A^{2n+1}.$$

Theorem 7.20. *Suppose that $\{c_n\}_{n=1}^\infty$ is a sequence in $\mathbb{C}$ and $R := \left(\limsup_{n\to\infty} |c_n|^{1/n}\right)^{-1}$. If $A \in \mathbb{C}^{J_N \times J_N}$ with $\|A\| < R$, then*

$$f(A) := \sum_{n=0}^\infty c_n A^n$$

is convergent and $f : B_0(R) := \{A : \|A\| < R\} \to \mathbb{C}^{J_N \times J_N}$ is continuous. Moreover the sum converges absolutely and uniformly on $B_0(\rho)$.

Proof. Let $\rho \in (\|A\|, R)$ and let $M_n := |c_n| \rho^n$. Then $\limsup_{n\to\infty} M_n^{1/n} = \rho/R < 1$ and therefore by the Root test, $\sum_{n=0}^\infty M_n < \infty$. Since, for $n \ge 1$,

$$\|c_n A^n\| = |c_n| \|A^n\| \le |c_n| \|A\|^n \le M_n,$$

the result is now again a consequence of the follows ■

Theorem 7.21 (Matrix Exponentials and etc.). *The series for* e^A, $\sin(A)$, $\cos(A)$, $\sinh(A)$, *and* $\cosh(A)$ *are all absolutely convergent define continuous functions from* $\mathbb{C}^{J_N \times J_N}$ *to* $\mathbb{C}^{J_N \times J_N}$ *in the Hilbert Schmidt norm.*

Proof. Since the corresponding numerical series all of infinite radius of convergence the result follows from Theorem 7.20.

■

Exercise 7.7 (Inverting perturbations of the identity). For $\|A\|_2 < 1$ in $\mathbb{C}^{J_N \times J_N}$, $I - A$ is invertible and

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n$$

where the sum is absolutely convergent. Moreover the function $A \to (I - A)^{-1}$ is continuous on the ball, $B := \{A : \|A\|_2 < 1\}$ and

$$\left\|(I - A)^{-1}\right\|_2 \leq \sum_{n=0}^{\infty} \|A^n\|_2 \leq \frac{\|A\|_2}{1 - \|A\|_2} + \sqrt{N}.$$