Math 150B Homework Problems: Winter 2020

Problems are our text book, "Differential Forms," by Guillemin and Haine or from the lecture notes as indicated. The problems from the lecture notes are restated here, however there may be broken references. If this is the case, please find the corresponding problem in the lecture notes for the proper references and for more context of the problem.

1.0 Homework 0, Due Wednesday, January 8, 2020 (Not to be collected)

- Lecture note Exercises: 2.1, 2.2, 2.3, 2.4, and 2.5.

1.1 Homework 1. Due Thursday, January 16, 2020

- Lecture note Exercises: 2.6, 2.7, 2.8, 2.11
- Book Exercises: 1.2.vi.

1.2 Homework 2. Due Thursday, January 23, 2020

- Lecture note Exercises: 2.9, 2.10, 2.12, 2.13, 2.14, 2.15, 2.16
- Book Exercises: 1.3.iii., 1.3.v., 1.3.vii, 1.4ix

1.3 Homework 3. Due Thursday, January 30, 2020

- Lecture note Exercises: 2.20, 2.21, 2.24, 2.25, 2.26, 2.27, 2.28
- Look at (but **don't hand in**) Exercises 2.22, 2.23 and the Book Exercises: 1.7.iv., 1.8vi.

1.4 Homework 4. Due Thursday, February 6, 2020

These problems are part of your midterm and are to be worked on by your-self. These are due at the start of the in-class portion of the midterm which is in class on **Thursday February 6, 2020.**

- Lecture note Exercises: 2.17, 2.18, 2.19, 2.29, 2.30

1.5 Homework 5. Due Thursday, February 13, 2020

- Lecture note Exercises: 2.31, 2.32, 2.33, 2.34, 2.35, 2.36
- Book Exercises: 2.3.ii., 2.3.iii., 2.4.i

1.6 Homework 6. Due Thursday, February 20, 2020

- Book Exercises: 2.1vii, 2.1viii, 2.4.ii, 2.4iii, 2.4iv. 2.6i, 2.6ii, 2.6iii (Refer to exercise 2.1.vii not 2.2viii), 3.2.i, 3.2viii
- Have a look at Reyer Sjamaar's notes: Manifolds and Differential Forms – especially see Chapter 6 starting on page 75 for the notions of a manifold, tangent spaces, and lots of pictures!

1.7 Homework 7. Now Due Friday, February 28, 2020 at 7:00PM

- Hand in Lecture note Exercises: 2.37 (now corrected), 2.38, 2.39, 2.40, 2.41, 2.42, 2.43, 2.44, 2.46, 2.47
- Look at but do not turn in Lecture note Exercise: 2.48

1.8 Homework 8. Due Friday, March 6, 2020 at 7:00PM

- Hand in Lecture note Exercises: 2.49, 2.50, 2.51, 2.52
- Look at but do not turn in Lecture note Exercise: 2.45 (done in class!)

Lecture Note Problems

Exercise 2.1. Find the value of the following integral;

$$I := \int_1^9 dy \int_{\sqrt{y}}^3 dx\ xe^y.$$

Hint: use Tonelli's theorem to change the order of integrations.

Exercise 2.2. Write the following iterated integral

$$I := \int_0^1 dx \int_{y=x^{2/3}}^1 dy\ xe^{y^4}.$$

as a multiple integral and use this to change the order of integrations and then compute I.

Exercise 2.3. Suppose that $d = 2$, show $m_2(B(0,r)) = \pi r^2$.

Exercise 2.4. Suppose that $d = 3$, show $m_3(B(0,r)) = \frac{4\pi}{3} r^3$.

Exercise 2.5. Let $V_d(r) := m_d(B(0,r))$. Show for $d \geq 1$ that

$$V_{d+1}(r) = \int_{-r}^r dz \cdot V_d\left(\sqrt{r^2 - z^2}\right) = r \int_{-\pi/2}^{\pi/2} V_d(r\cos\theta)\cos\theta d\theta.$$

Exercise 2.6 (Change of variables for elementary matrices). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. Show by direct calculation that;

$$|\det T| \int_{\mathbb{R}^d} f(T(x))\, dx = \int_{\mathbb{R}^d} f(y)\, dy \tag{2.1}$$

for each of the following linear transformations;

1. Suppose that $i < k$ and

$$T(x_1, x_2 \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_k, x_{i+1} \ldots, x_{k-1}, x_i, x_{k+1}, \ldots x_d),$$

   i.e. T swaps the i and k coordinates of x. [In matrix notation T is the identity matrix with the i and k column interchanged.]
2. $T(x_1, \ldots x_k, \ldots, x_d) = (x_1, \ldots, cx_k, \ldots x_d)$ where $c \in \mathbb{R} \setminus \{0\}$. [In matrix notation, $T = [e_1| \ldots |e_{k-1}|ce_k|e_{k+1}| \ldots |e_d]$.]
3. $T(x_1, x_2 \ldots, x_d) = (x_1, \ldots, \overset{\text{i'th spot}}{x_i + cx_k}, \ldots x_k, \ldots x_d)$ where $c \in \mathbb{R}$. [In matrix notation $T = [e_1| \ldots |e_i| \ldots |e_k + ce_i|e_{k+1}| \ldots |e_d]$.

Hint: you should use Fubini's theorem along with the one dimensional change of variables theorem.

[To be more concrete here are examples of each of the T appearing above in the special case $d = 4$,

1. If $i = 2$ and $k = 3$ then $T = \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix}$.
2. If $k = 3$ then $T = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&c&0 \\ 0&0&0&1 \end{bmatrix}$,
3. If $i = 2$ and $k = 4$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + cx_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&c \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

while if $i = 4$ and $k = 2$,

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 + cx_2 \end{pmatrix} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&c&0&1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Exercise 2.7. Let $V = \mathbb{R}^n$ and $\beta = \{u_j\}_{j=1}^n$ be a basis for $\mathbb{R}^n$. Recall that every $\ell \in (\mathbb{R}^n)^*$ is of the form $\ell_a(x) = a \cdot x$ for some $a \in \mathbb{R}^n$. Thus the dual basis, β^*, to β can be written as $\left\{u_j^* = \ell_{a_j}\right\}_{j=1}^n$ for some $\{a_j\}_{j=1}^n \subset \mathbb{R}^n$. In this problem you are asked to show how to find the $\{a_j\}_{j=1}^n$ by the following steps.

1. Show that for $j \in [n]$, a_j must solve the following k-linear equations;

$$\delta_{j,k} = \ell_{a_j}(u_k) = a_j \cdot u_k = u_k^{\text{tr}} a_j \text{ for } k \in [n]. \tag{2.2}$$

2. Let $U := [u_1|\dots|u_n]$ (i.e. the columns of U are the vectors from β). Show that the equations in (2.2) may be written in matrix form as, $U^{\mathrm{tr}} a_j = e_j$, where $\{e_j\}_{j=1}^n$ is the standard basis for $\mathbb{R}^n$.
3. Conclude that $a_j = [U^{\mathrm{tr}}]^{-1} e_j$ or equivalently;

$$[a_1|\dots|a_n] = \left[U^{\mathrm{tr}}\right]^{-1}$$

Exercise 2.8. Let $V = \mathbb{R}^2$ and $\beta = \{u_1, u_2\}$, where

$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Find $a_1, a_2 \in \mathbb{R}^2$ explicitly so that explicitly the dual basis $\beta^* := \{u_1^* = \ell_{a_1}, u_2^* = \ell_{a_2}\}$ is the dual basis to β. Please explicitly verify your answer is correct by showing $u_j^*(u_k) = \delta_{jk}$.

Exercise 2.9. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. Show $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set if and only if $\{a_j\}_{j=1}^k \subset V$ is a linearly independent set.

Exercise 2.10. Let $V = \mathbb{R}^n$, $\{a_j\}_{j=1}^k \subset V$, and $\ell_j(x) = a_j \cdot x$ for $x \in \mathbb{R}^n$ and $j \in [k]$. If $\{\ell_j\}_{j=1}^k \subset V^*$ is a linearly independent set, show there exists $\{u_j\}_{j=1}^k \subset V$ so that $\ell_i(u_j) = \delta_{ij}$ for $i, j \in [k]$. Here is a possible outline.

1. Using Exercise 2.9 and citing a basic fact from Linear algebra, you may choose $\{a_j\}_{j=k+1}^n \subset V$ so that $\{a_j\}_{j=1}^n$ is a basis for V.
2. Argue that it suffices to find $u_j \in V$ so that

$$a_i \cdot u_j = \delta_{ij} \text{ for all } i, j \in [n]. \tag{2.3}$$

3. Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{R}^n$ and $A := [a_1|\dots|a_n]$ be the $n \times n$ matrix with columns given by that $\{a_j\}_{j=1}^n$. Show that the Eqs. (2.3) may be written as

$$A^{\mathrm{tr}} u_j = e_j \text{ for } j \in [n]. \tag{2.4}$$

4. Cite basic facts from linear algebra to explain why $A := [a_1|\dots|a_n]$ and A^{tr} are both invertible $n \times n$ matrices.
5. Argue that Eq. (2.4) has a unique solution, $u_j \in \mathbb{R}^n$, for each j.

Exercise 2.11. In this problem, let

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \text{ and } w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Which of the following functions formulas for T define a 2-tensors on $\mathbb{R}^3$. Please justify your answers.

1. $T(v, w) = v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1$.
2. $T(v, w) = v_1 + 7 v_1 + v_2$.
3. $T(v, w) = v_1^2 w_3 + v_2 w_1$,
4. $T(v, w) = \sin(v_1 w_3 + v_1 w_2 + v_2 w_1 + 7 v_1 w_1)$.

Exercise 2.12. If $T \in \Lambda^k(V^*)$, show $T(v_1, \dots, v_k) = 0$ whenever $\{v_i\}_{i=1}^k \subset V$ are linearly dependent.

Exercise 2.13. Let V, W, and Z be three finite dimensional vector spaces and suppose that $V \xrightarrow{T} W \xrightarrow{S} Z$ are linear transformations. Noting that $V \xrightarrow{ST} Z$, show $(ST)^* = T^* S^*$.

Exercise 2.14. If $\psi \in \Lambda^n(V^*) \setminus \{0\}$, show $\psi(v_1, \dots, v_n) \neq 0$ whenever $\{v_i\}_{i=1}^n \subset V$ are linearly independent. [Coupled with Exercise 2.12, it follows that $\psi(v_1, \dots, v_n) \neq 0$ iff $\{v_i\}_{i=1}^n \subset V$ are linearly independent.]

Exercise 2.15. Let $\{e_i\}_{i=1}^4$ be the standard basis for $\mathbb{R}^4$ and $\{\varepsilon_i = e_i^*\}_{i=1}^4$ be the associated dual basis (i.e. $\varepsilon_i(v) = v_i$ for all $v \in \mathbb{R}^4$.) Compute;

1. $\varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_4 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \right)$,
2. $\varepsilon_3 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$,
3. $\varepsilon_1 \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$,
4. $(\varepsilon_1 + \varepsilon_3) \wedge \varepsilon_2 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right)$, and
5. $\varepsilon_4 \wedge \varepsilon_3 \wedge \varepsilon_2 \wedge \varepsilon_1 (e_1, e_2, e_3, e_4)$.

Exercise 2.16. Show, using basic knowledge of determinants, that for $\ell_0, \ell_1, \ell_2, \ell_3 \in V^*$, that

$$(\ell_0 + \ell_1) \wedge \ell_2 \wedge \ell_3 = \ell_0 \wedge \ell_2 \wedge \ell_3 + \ell_1 \wedge \ell_2 \wedge \ell_3.$$

Exercise 2.17. Suppose $\{\ell_j\}_{j=1}^k \subset [\mathbb{R}^n]^*$.

1. Explaining why $\ell_1 \wedge \cdots \wedge \ell_k = 0$ if $\ell_i = \ell_j$ for some $i \neq j$.
2. Show $\ell_1 \wedge \cdots \wedge \ell_k = 0$ if $\{\ell_j\}_{j=1}^k$ are linear **dependent.** [You may assume that $\ell_1 = \sum_{j=2}^k a_j \ell_j$ for some $a_j \in \mathbb{R}$.]

Exercise 2.18. If $\{\ell_j\}_{j=1}^k \subset [\mathbb{R}^n]^*$ are linearly **independent**, show

$$\ell_1 \wedge \cdots \wedge \ell_k \neq 0.$$

Hint: make use of Exercise 2.10.

Exercise 2.19. Let $\{\varepsilon_j\}_{j=1}^3$ be the standard dual basis and $v = (1,2,3)^{\mathrm{tr}} \in \mathbb{R}^3$, find $a_1, a_2, a_3 \in \mathbb{R}$ so that

$$i_v(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3) = a_1 \varepsilon_2 \wedge \varepsilon_3 + a_2 \varepsilon_1 \wedge \varepsilon_3 + a_3 \varepsilon_1 \wedge \varepsilon_2.$$

Exercise 2.20 (Cross I). For $a \in \mathbb{R}^3$, let $\ell_a(v) = a \cdot v = a^{\mathrm{tr}} v$, so that $\ell_a \in (\mathbb{R}^3)^*$. In particular we have $\varepsilon_i = \ell_{e_i}$ for $i \in [3]$ is the dual basis to the standard basis $\{e_i\}_{i=1}^3$. Show for $a, b \in \mathbb{R}^3$,

$$\ell_a \wedge \ell_b = i_{a \times b}[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3] \tag{2.5}$$

Hints: 1) write $\ell_a = \sum_{i=1}^3 a_i \varepsilon_i$ and 2) make use of Eq. (**??**)

Exercise 2.21 (Cross II). Use Exercise 2.20 to prove the standard vector calculus identity;

$$(a \times b) \cdot (x \times y) = (a \cdot x)(b \cdot y) - (b \cdot x)(a \cdot y)$$

which is valid for all $a, b, x, y \in \mathbb{R}^3$.Hint: evaluate Eq. (2.5) at (x, y) while using Lemma **??**.

Exercise 2.22 (Surface Integrals). In this exercise, let $\omega \in \mathcal{A}_3(\mathbb{R}^3)$ be the standard volume form, $\omega(v_1, v_2, v_3) := \det[v_1|v_2|v_3]$, suppose D is an open subset of $\mathbb{R}^2$, and $\Sigma : D \to S \subset \mathbb{R}^3$ is a "parameterized surface," refer to Figure 2.1. If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field on $\mathbb{R}^3$, then from your vector calculus class,

$$\iint_S F \cdot N dA = \varepsilon \cdot \iint_D F(\Sigma(u,v)) \cdot [\Sigma_u(u,v) \times \Sigma_v(u,v)]\, du dv \tag{2.6}$$

where $\varepsilon = 1$ ($\varepsilon = -1$) if $N(\Sigma(u,v))$ points in the same (opposite) direction as $\Sigma_u(u,v) \times \Sigma_v(u,v)$. We assume that ε is independent of $(u,v) \in D$.

Show the formula in Eq. (2.6) may be rewritten as

$$\iint_S F \cdot N dA = \varepsilon \iint_D \left(i_{F(\Sigma(u,v))}\omega\right)(\Sigma_u(u,v), \Sigma_v(u,v))\, du dv \tag{2.7}$$

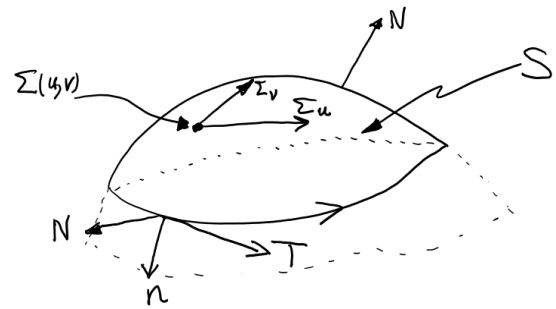

Fig. 2.1. In this figure N is a smoothly varying normal to S, n is a normal to the boundary of S, and T is a tangential vector to the boundary of S. Moreover, $D \ni (u,v) \to \Sigma(u,v) \in S$ is a parametrization of S where $D \subset \mathbb{R}^2$.

where

$$\varepsilon := \operatorname{sgn}(\omega(N \circ \Sigma, \Sigma_u, \Sigma_v)) = \begin{cases} 1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) > 0 \\ -1 & \text{if } \omega(N \circ \Sigma, \Sigma_u, \Sigma_v) < 0. \end{cases}$$

Remarks: Once we introduce the proper notation, we will be able to write Eq. (2.7) more succinctly as

$$\iint_S F \cdot N dA = \iint_S i_F \omega := \varepsilon \iint_D \Sigma^*(i_F \omega).$$

Exercise 2.23 (Boundary Orientation). Referring to the set up in Exercise 2.22, the tangent vector T has been chosen by using the "right-hand" rule in order to determine the orientation on the boundary, ∂S, of S so that Stoke's theorem holds, i.e.

$$\iint_S [\nabla \times F] \cdot N dA = \int_{\partial S} F \cdot T ds. \tag{2.8}$$

Show by using the "right hand rule" that $T = c \cdot N \times n$ with $c > 0$ and then also show

$$c = \omega(N, n, T) = (i_n i_N \omega)(T).$$

Also note by Exercise 2.22, that Eq. (2.8) may be written as

$$\iint_S i_{\nabla \times F} \omega = \int_{\partial S} F \cdot T ds \tag{2.9}$$

Remark: We will introduce the "one form", $F \cdot dx$ and an "exterior derivative" operator, d, so that

$$d\left[F\cdot dx\right]=i_{\nabla\times F}\omega$$

and Eq. (2.9) may be written in the pleasant form,

$$\iint_S d\left[F\cdot dx\right]=\int_{\partial S}F\cdot dx.$$

Exercise 2.24. Let

$$f\begin{pmatrix}r\\ \theta\end{pmatrix}=\begin{bmatrix}r\cos\theta\\ r\sin\theta\end{bmatrix}\text{ for }\begin{pmatrix}r\\ \theta\end{pmatrix}\in\mathbb{R}^2.$$

Find;

$$f'\begin{pmatrix}r\\ \theta\end{pmatrix}\text{ and then show }\det\left[f'\begin{pmatrix}r\\ \theta\end{pmatrix}\right]=r.$$

Exercise 2.25. Let

$$f\begin{pmatrix}r\\ \theta\\ \varphi\end{pmatrix}=\begin{bmatrix}r\sin\varphi\cdot\cos\theta\\ r\sin\varphi\cdot\sin\theta\\ r\cos\varphi\end{bmatrix}\text{ for }\begin{pmatrix}r\\ \theta\\ \varphi\end{pmatrix}\in\mathbb{R}^3.$$

Find;

$$f'\begin{pmatrix}r\\ \theta\\ \varphi\end{pmatrix}\text{ and then show }\det\left[f'\begin{pmatrix}r\\ \theta\\ \varphi\end{pmatrix}\right]=-r^2\sin\varphi.$$

Exercise 2.26. Let

$$A=\begin{bmatrix}A_{11}&A_{12}&\dots&A_{1n}\\ A_{21}&A_{22}&\dots&A_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ A_{n1}&A_{n2}&\dots&A_{nn}\end{bmatrix}=\left[a_1|\dots|a_n\right]$$

be an $n\times n$ matrix with i^{th}-column

$$a_i=\begin{bmatrix}A_{1i}\\ A_{2i}\\ \vdots\\ A_{ni}\end{bmatrix}.$$

Given another $n\times n$ matrix B with analogous notation, show

$$\left(\partial_B\det\right)(A)=\sum_{j=1}^n\det\left[a_1|\dots|a_{j-1}|b_j|a_{j+1}|\dots|b_n\right].\tag{2.10}$$

For example if $n=3$, this formula reads,

$$\left(\partial_B\det\right)(A)=\det\left[b_1|a_2|a_3\right]+\det\left[a_1|b_2|a_3\right]+\det\left[a_1|a_2|b_3\right].$$

Suggestions; by definition,

$$\left(\partial_B\det\right)(A):=\frac{d}{dt}|_0\det\left(A+tB\right)=\frac{d}{dt}|_0\det\left[a_1+tb_1|\dots|a_n+tb_n\right].$$

Now apply Lemma **??** with

$$f\left(x_1,\dots,x_n\right)=\det\left[a_1+x_1b_1|\dots|a_n+x_nb_n\right].$$

Exercise 2.27 (Exercise 2.26 continued). Continuing the notation and results from Exercise 2.26, show;

1. If $A=I$ is the $n\times n$ identity matrix in Eq. (2.10), then

$$\left(\partial_B\det\right)(I)=\operatorname{tr}(B)=\sum_{j=1}^nB_{j,j}.$$

2. If A is an $n\times n$ invertible matrix, shows

$$\left(\partial_B\det\right)(A)=\det(A)\cdot\operatorname{tr}\left(A^{-1}B\right).$$

Hint: Verify the identity,

$$\det\left(A+tB\right)=\det(A)\cdot\det\left(I+tA^{-1}B\right)$$

which you should then use along with first item of this exercise.

Exercise 2.28. Using Proposition **??**, find df when

$$f\left(x_1,x_2,x_3\right)=x_1^2\sin\left(e^{x_2}\right)+\cos\left(x_3\right).$$

Exercise 2.29. Let $g_1,g_2,\dots,g_n\in C^1\left(U,\mathbb{R}\right)$, $f\in C^1\left(\mathbb{R}^n,\mathbb{R}\right)$, and $u=f\left(g_1,\dots,g_n\right)$, i.e.

$$u(p)=f\left(g_1(p),\dots,g_n(p)\right)\text{ for all }p\in U.$$

Show

$$du=\sum_{j=1}^n\left(\partial_jf\right)\left(g_1,\dots,g_n\right)dg_j$$

which is to be interpreted to mean,

$$du\left(v_p\right)=\sum_{j=1}^n\left(\partial_jf\right)\left(g_1(p),\dots,g_n(p)\right)dg_j\left(v_p\right)\text{ for all }v_p\in TU.$$

Hint: For $v_p\in TU$, let $\sigma(t)=\left(g_1(p+tv),\dots,g_n(p+tv)\right)$ and then make use of the chain rule (see Eq. (**??**)) to compute $du\left(v_p\right)$.

Exercise 2.30 (Chain Rule for Maps). Suppose that $f : U \to V$ and $g : V \to W$ are C^1-functions where U, V, and W are open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^p$ respectively and let $g \circ f : U \to W$ be the composition map,

$$g \circ f : U \xrightarrow{f} V \xrightarrow{g} W.$$

Show

$$(g \circ f)'(p) = g'(f(p)) f'(p) \text{ for all } p \in U. \tag{2.11}$$

Hint: Let $v \in \mathbb{R}^n$ and $\sigma(t) := f(p + tv)$ – a differentiable curve in V. Then use the chain rule in Theorem **??** twice in order to compute,

$$(g \circ f)'(p) v = \frac{d}{dt}|_0 g(f(p + tv)) = \frac{d}{dt}|_0 g(\sigma(t)).$$

Exercise 2.31. Suppose that $\{x_j\}_{j=1}^4$ are the standard coordinates on $\mathbb{R}^4$, $p = (1, -1, 2, 3)^{\mathrm{tr}} \in \mathbb{R}^4$, $v^1 = (1, 2, 3, 4)^{\mathrm{tr}}$, $v^2 = (0, 1, -1, 1)^{\mathrm{tr}}$, $v^3 = (1, 0, 3, 2)$,

$$\alpha = x_4 (dx_1 + dx_2), \ \beta = x_1 x_2 (dx_3 + dx_4), \text{ and } \omega = \left(x_1^2 + x_3^2\right) dx_3 \wedge dx_2 \wedge dx_4.$$

Compute the following quantities;

1. $\alpha\left(v_p^1\right)$,
2. $\alpha \wedge \alpha\left(v_p^1, v_p^2\right)$,
3. $\alpha \wedge \beta\left(v_p^1, v_p^2\right)$,
4. $\omega\left(v_p^1, v_p^2, v_p^3\right)$.

Exercise 2.32. Let $\{x_i\}_{i=1}^6$ be the standard coordinates on $\mathbb{R}^6$ and let

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_5 \wedge dx_6 \in \Omega^2\left(\mathbb{R}^6\right).$$

Show

$$\omega \wedge \omega \wedge \omega = c dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \wedge dx_6,$$

for some $c \in \mathbb{R}$ which you should find.

Exercise 2.33. Let $\alpha = x dx - y dy$, $\beta = z dx \wedge dy + x dy \wedge dz$ and $\gamma = z dy$ on $\mathbb{R}^3$, calculate,

$$\alpha \wedge \beta, \quad \alpha \wedge \beta \wedge \gamma, \quad d\alpha, \quad d\beta, \quad d\gamma.$$

Exercise 2.34. Let (x, y) be the standard coordinates on $\mathbb{R}^2$, and define,

$$\alpha := \left(x^2 + y^2\right)^{-1} \cdot (x dy - y dx) \in \Omega^1\left(\mathbb{R}^2 \setminus \{0\}\right).$$

Show α is closed. [We will eventually see that this form is not exact.]

Exercise 2.35 (Divergence Formula). Let $f = (f_1, f_2, f_3, \ldots, f_n)$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$. By Example **??** with $k = n$ we have

$$i_f \omega = i_F \omega = \sum_{j=1}^n (-1)^{j-1} f_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$

Show

$$d[i_F \omega] = (\nabla \cdot f) \omega \text{ where } \nabla \cdot f = \sum_{i=1}^n \partial_i f_i,$$

i.e. $\nabla \cdot f$ is the divergence of f from your vector calculus course.

Exercise 2.36 (Curl Formula). Let $f = (f_1, f_2, f_3) \in C^\infty\left(\mathbb{R}^3, \mathbb{R}^3\right)$,

$$\begin{aligned} \omega &= dx_1 \wedge dx_2 \wedge dx_3, \text{ and} \\ \alpha &= f \cdot (dx_1, dx_2, dx_3) := f_1 dx_1 + f_2 dx_2 + f_3 dx_3. \end{aligned}$$

Show $d\alpha = i_{\nabla \times f} \omega$ where $\nabla \times f$ is the usual vector calculus curl of f, see Eq. (**??**) of Definition **??** with F replaced by $f = (f_1, f_2, f_3)$.

Exercise 2.37. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the smooth map,

$$f(x, y) = \left(x^2 - y^2, 2xy\right).$$

Show $\|f(x, y)\| \to \infty$ when $\|(x, y)\| \to \infty$ which turns out to be equivalent to the statement the f is proper, see Example **??** below. Further compute the $\deg(f)$.

Exercise 2.38. Show that x in Definition **??** is necessarily unique by showing $d(x, y) = 0$ if $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Exercise 2.39. Consider $\mathbb{N}$ as a metric space with $d(m, n) := |m - n|$ and suppose that (Y, d) is a metric space. Show that every function, $f : \mathbb{N} \to Y$ is continuous.

Exercise 2.40. Suppose that (X, d) is a metric space and $f, g : X \to \mathbb{C}$ are two continuous functions on X. Show;

1. $f + g$ is continuous,
2. $f \cdot g$ is continuous,
3. f/g is continuous provided $g(x) \neq 0$ for all $x \in X$.

Exercise 2.41. (You only need explain parts 1. and 5. of this problem as we have done the other parts in class.) Show the following functions from $\mathbb{C}$ to $\mathbb{C}$ are continuous.

1. $f(z) = c$ for all $z \in \mathbb{C}$ where $c \in \mathbb{C}$ is a constant.
2. $f(z) = |z|$. [**Hint:** $f(z) = d(z, 0)$ where $d(z, w) := |z - w|$ is the Euclidean norm on $\mathbb{C} \cong \mathbb{R}^2$.]
3. $f(z) = z$ and $f(z) = \bar{z}$.
4. $f(z) = \operatorname{Re} z$ and $f(z) = \operatorname{Im} z$.
5. $f(z) = \sum_{m,n=0}^{N} a_{m,n} z^m \bar{z}^n$ where $a_{m,n} \in \mathbb{C}$.

Exercise 2.42. Prove item 2. of Theorem **??**. If $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed subsets of X, then $\cap_{\alpha \in I} C_\alpha$ is closed in X.

Exercise 2.43. Let (X, d) be a metric space and $C := \{x_1, \ldots, x_n\}$ be a finite subset of X. Show C is closed and hence $X \setminus C$ is an open.

Exercise 2.44. Give an example of a collection of closed subsets, $\{A_n\}_{n=1}^{\infty}$, of $\mathbb{C}$ such that $\cup_{n=1}^{\infty} A_n$ is not closed.

Exercise 2.45. Suppose that K and F are compact subsets of a metric space, (X, d). Show $K \cup F$ is compact in X as well.

Exercise 2.46. Explain why;

1. $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = 1/x$ is a proper map.
2. $f : (0, \infty) \to (-\infty, \infty)$ defined by $f(x) = 1/x$ is **not** a proper map.

Exercise 2.47. Let $V := \mathbb{R}^2 \setminus \{0\}$ and $f : V \to V$ be defined by $f(x) = \frac{x}{\|x\|^2}$ for $x = (x_1, x_2) \in \mathbb{R}^2$.

1. Explain why f is a proper map.
2. Show $f \circ f(x) = x$ so that f is in fact a homeomorphism.
3. Compute the $\deg(f)$. [Hint: evaluate $f'(p)$ at your favorite point in V.]

Exercise 2.48. Let

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0, \end{cases}$$

see Figure 2.2. Show $f \in C^\infty(\mathbb{R}, [0, 1])$. Here is a possible outline.

1. First show for $x > 0$ and $n \in \mathbb{N}$ there is a polynomial function, $p_n(t)$, such that $f^{(n)}(x) = p_n(x^{-1}) f(x)$. For example $p_1(t) = t^2$ and $p_2(t) = t^4 - 2t^3$.
2. Show $\lim_{t \to \infty} (t^n e^{-t}) = 0$ for all $n \in \mathbb{N}$ (hint take logarithms) and then use this to show $\lim_{x \downarrow 0} f^{(n)}(x) = 0$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
3. Show by the mean value theorem along with induction on n that $f \in C^n(\mathbb{R}, \mathbb{R})$ and $f^{(k)}(0) = 0$ for $0 \le k \le n$ for all $n \in \mathbb{N}$.

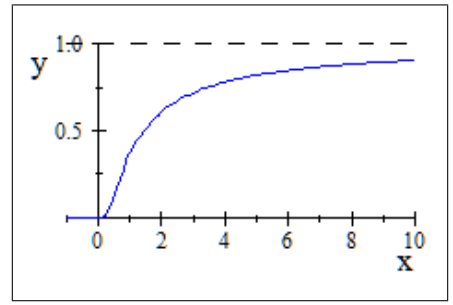

Fig. 2.2. Plot of $y = f(x)$.

Exercise 2.49. As mentioned in class, the study of complex variables is essentially the study of functions (or vector-fields if you prefer) $f \in C^1(U \to \mathbb{R}^2)$ (U is an open subset of $\mathbb{R}^2$) of the form $f(x, y) = (u(x, y), v(x, y))^{\mathrm{tr}}$ such that u and v satisfy the **Cauchy Riemann equations;**

$$u_y = -v_x \text{ and } v_v = u_x \text{ on } U. \tag{2.12}$$

Show under these assumptions that

$$\det f' = u_x^2 + u_y^2 = v_x^2 + v_y^2 \ge 0 \text{ on } U.$$

Exercise 2.50. Using Eq. (**??**) (justified by Corollary **??**) show; if $g : \mathbb{R}^2 \to \mathbb{R}$ is a bounded compactly supported function, then

$$\int_{\mathbb{R}^2} g(x^2 - y^2, 2xy)\, 4(x^2 + y^2)\, dx \wedge dy = 2 \cdot \int_{\mathbb{R}^2} g(u, v)\, du \wedge dv.$$

Hint: make the "change of variables" $(u, v) = f(x, y) = (x^2 - y^2, 2xy)$ and use your results from Exercises 2.37.

Exercise 2.51. Use Exercises 2.50 to find the explicit value for the following integral,

$$\int_{\mathbb{R}^2} 1_{(1,2)}(x^2 - y^2) \cdot 1_{(0,1)}(2xy) \cdot (x^4 - y^4)\, dx \wedge dy.$$

Hint:

$$(x^4 - y^4) = (x^2 - y^2)(x^2 + y^2) = u(x^2 + y^2).$$

Exercise 2.52. Let D be a connected open subset of $\mathbb{R}^k$, U be an open subset of $\mathbb{R}^n$, $\gamma : D \to U$ be a parameterized k-surface in U as in Definition **??**. Further assume that $\omega \in \Omega^k(U)$ is such that $\gamma^* \omega \in \Omega_c^k(D)$. If Q is another connected open subset of $\mathbb{R}^k$ and $\varphi : Q \to D$ is a diffeomorphism, show

$$\int_{\gamma\circ\varphi} \omega = \deg(\varphi) \cdot \int_{\gamma} \omega$$

where in this case $\deg(\varphi) \in \{\pm 1\}$.