Math 180C Homework Problems

0.1 Homework #1 (Due Monday, April 7)

Exercise 0.1 (2nd order recurrence relations). Let a, b, c be real numbers and suppose that $\{y_n\}_{n=-\infty}^{\infty}$ solves the second order homogeneous recurrence relation:

$$a y_{n+1} + b y_n + c y_{n-1} = 0. \tag{0.1}$$

Show:

1. for any $\lambda \in \mathbb{C}$,
$$a\lambda^{n+1} + b\lambda^n + c\lambda^{n-1} = \lambda^{n-1} p(\lambda) \tag{0.2}$$
where $p(\lambda) = a\lambda^2 + b\lambda + c$.
2. Let $\lambda_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ be the roots of p and suppose for the moment that $b^2 - 4ac \neq 0$. Show
$$y_n := A_+ \lambda_+^n + A_- \lambda_-^n$$
solves Eq. (0.1) for any choice of A_+ and A_-.
3. Now suppose that $b^2 = 4ac$ and $\lambda_0 := -b/(2a)$ is the double root of $p(\lambda)$. Show that
$$y_n := (A_0 + A_1 n)\lambda_0^n$$
solves Eq. (0.1) for any choice of A_0 and A_1. **Hint:** Differentiate Eq. (0.2) with respect to λ and then set $\lambda = \lambda_0$.
4. Show that every solution to Eq. (0.1) is of the form found in parts 2. and 3.

In the next couple of exercises you are going to use first step analysis to show that a simple unbiased random walk on $\mathbb{Z}$ is null recurrent. We let $\{X_n\}_{n=0}^{\infty}$ be the Markov chain with values in $\mathbb{Z}$ with transition probabilities given by

$$P(X_{n+1} = j \pm 1 | X_n = j) = 1/2 \text{ for all } n \in \mathbb{N}_0 \text{ and } j \in \mathbb{Z}.$$

Further let $a, b \in \mathbb{Z}$ with $a < 0 < b$ and

$$T_{a,b} := \min\{n : X_n \in \{a, b\}\} \text{ and } T_b := \inf\{n : X_n = b\}.$$

We know by Proposition 2.13 that $\mathbb{E}_0[T_{a,b}] < \infty$ from which it follows that $P(T_{a,b} < \infty) = 1$ for all $a < 0 < b$.

Exercise 0.2. Let $w_j := P_j\left(X_{T_{a,b}} = b\right) := P\left(X_{T_{a,b}} = b|X_0 = j\right).$

1. Use first step analysis to show for $a < j < b$ that
$$w_j = \frac{1}{2}\left(w_{j+1} + w_{j-1}\right) \tag{0.3}$$
provided we define $w_a = 0$ and $w_b = 1$.
2. Use the results of Exercise 0.1 to show
$$P_j\left(X_{T_{a,b}} = b\right) = w_j = \frac{1}{b-a}(j-a). \tag{0.4}$$
3. Let
$$T_b := \begin{cases} \min\{n : X_n = b\} & \text{if} \quad \{X_n\} \text{ hits } b \\ \infty & \text{otherwise} \end{cases}$$
be the first time $\{X_n\}$ hits b. Explain why, $\left\{X_{T_{a,b}} = b\right\} \subset \{T_b < \infty\}$ and use this along with Eq. (0.4) to conclude that $P_j(T_b < \infty) = 1$ for all $j < b$. (By symmetry this result holds true for all $j \in \mathbb{Z}$.)

Exercise 0.3. The goal of this exercise is to give a second proof of the fact that $P_j(T_b < \infty) = 1$. Here is the outline:

1. Let $w_j := P_j(T_b < \infty)$. Again use first step analysis to show that w_j satisfies Eq. (0.3) for all j with $w_b = 1$.
2. Use Exercise 0.1 to show that there is a constant, c, such that
$$w_j = c(j-b) + 1 \text{ for all } j \in \mathbb{Z}.$$
3. Explain why c must be zero to again show that $P_j(T_b < \infty) = 1$ for all $j \in \mathbb{Z}$.

Exercise 0.4. Let $T = T_{a,b}$ and $u_j := \mathbb{E}_j T := \mathbb{E}[T|X_0 = j]$.

1. Use first step analysis to show for $a < j < b$ that
$$u_j = \frac{1}{2}\left(u_{j+1} + u_{j-1}\right) + 1 \tag{0.5}$$
with the convention that $u_a = 0 = u_b$.
2. Show that
$$u_j = A_0 + A_1 j - j^2 \tag{0.6}$$
solves Eq. (0.5) for any choice of constants A_0 and A_1.
3. Choose A_0 and A_1 so that u_j satisfies the boundary conditions, $u_a = 0 = u_b$. Use this to conclude that
$$\mathbb{E}_j T_{a,b} = -ab + (b+a)j - j^2 = -a(b-j) + bj - j^2. \tag{0.7}$$

Remark 0.1. Notice that $T_{a,b} \uparrow T_b = \inf\{n : X_n = b\}$ as $a \downarrow -\infty$, and so passing to the limit as $a \downarrow -\infty$ in Eq. (0.7) shows
$$\mathbb{E}_j T_b = \infty \text{ for all } j < b.$$

Combining the last couple of exercises together shows that $\{X_n\}$ is null - recurrent.