Math 21D (Driver) Midterm II Practice Test

1. Sketch the direction fields for the differential equation

$$y' = y^3 - y. \tag{1}$$

2. Suppose that y solves equation (1). Find the $\lim_{t\to\infty} y(t)$ if:

   (a) $y(0) = 0$.

   (b) $y(0) = .5$

   (c) $y(0) = 2$

   (d) $y(0) = -.5$.

3. Find the solution to the initial value problem,

$$(1-t^2)y'(t) - ty(t) = t(1-t^2) \text{ with } y(0) = 2.$$

4. Describe **implicit**ly the general solution to the differential equation

$$y'(x) = \frac{\sin(x)}{y^4 + y^2 + 2}.$$

   Do not try to solve explicitly for $y(x)$.

5. Determine which of the following differential equations is exact:

$$\begin{aligned}(1 + e^{xy}y)dx + xe^{xy}dy &= 0. &\text{(a)}\\ (y^2 + e^{xy}y)dx + xe^{xy}dy &= 0. &\text{(b)}\end{aligned}$$

6. For the equation above which is exact, find a function $\psi(x, y)$ so that any solution to the equation satisfies $\psi(x, y(x)) = C$, for some constant C.

7. Determine the largest interval in which the given initial value problem

$$\sin(t)y''(t) - \cos(t)y'(t) + 4t^3y(t) = e^t, \qquad y(6\pi/5) = 2, \qquad y'(6\pi/5) = 1.$$

   is certain to have a unique twice differentiable solution. (Do not solve the equation!!)

8. Show that te^t and e^t are two solutions to the differential equation

$$y''(t) - 2y'(t) + y(t) = 0.$$

   Use these solutions to find the solution y which satisfies the initial conditions $y(1) = 1$ and $y'(1) = -1$.

9. Compute the Wronskian $W(t)$ of $y_1(t) = e^t \sin(t)$ and $y_2(t) = e^{2t}$.

10. Let $L(y) = \ddot{y} - 8\dot{y} + 16y$. Find a particular solution to the differential equation $L(y) = g$ for a) $g(t) = 32t^2$ and for b) $g(t) = 2e^{4t}$.

Solution: a) Let $Y(t) = At^2 + Bt + C$, then

$$\begin{aligned} L(Y) &= A(2 - 16t + 16t^2) + B(-8 + 16t) + C\,16 \\ &= 16At^2 + 16t(B - A) + 2A - 8B + 16C \stackrel{\text{set}}{=} 32t^2. \end{aligned}$$

To solve this equation, we must take $A = 2$, $B = 2$, and C so that

$$0 = 2A - 8B + 16C = 4 - 16 + 16C.$$

That is $C = 12/16 = 3/4$. Hence $Y(t) = 2t^2 + 2t + 3/4$ is a particular solution.

b) Notice that

$$L(e^{\lambda t}) = (\lambda^2 - 8\lambda + 16)e^{\lambda t} = (\lambda - 4)^2 e^{\lambda t}. \tag{2}$$

Unfortunately, $p(\lambda) = (\lambda^2 - 8\lambda + 16) = (\lambda - 4)^2$ has $\lambda = 4$ a double root. So we must differentiate Eq. (2) twice with respect to λ to find

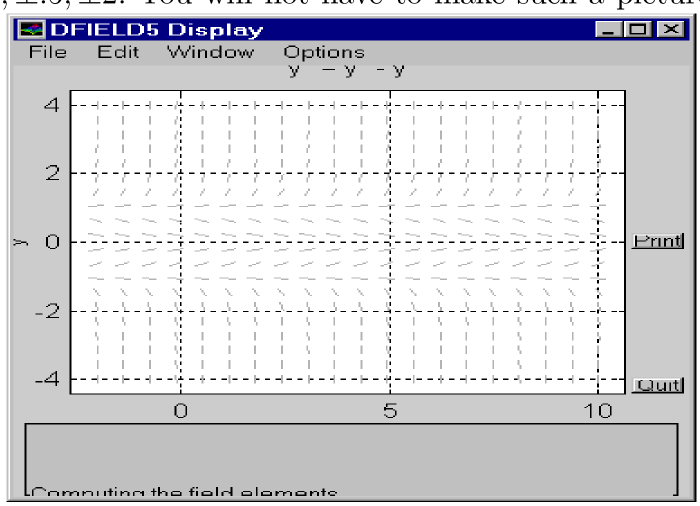
$$L(t^2 e^{4t}) = L(t^2 e^{\lambda t})|_{\lambda=4} = \frac{\partial^2}{\partial \lambda^2}|_{\lambda=4} \left\{(\lambda - 4)^2 e^{\lambda t}\right\} = 2e^{4t}.$$

So in fact $Y(t) = t^2 e^{4t}$ is a particular solution.

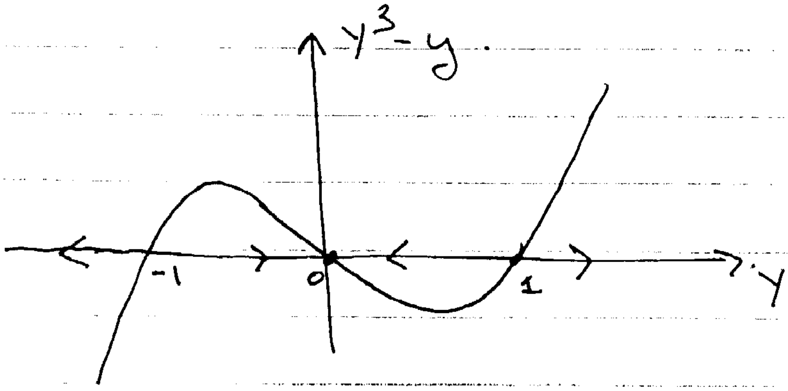
11. **Hint:** perhaps one of the word problems on your homework will appear on the test!

1 Solutions To Problems 1 – 9.

1) The key features are at $y = -1, 1, 0, \pm.5, \pm2$. You will not have to make such a picture but you should know how to make one.



#2



$\lim_{t\to\infty} y(t)$	$y(0)$
0	0
0	$\frac{1}{2}$
∞	2
0	$-\frac{1}{2}$

#3) $y' = \frac{t}{1-t^2} y + t$

Let $\mu(t) = e^{-\int \frac{t}{1-t^2} dt} = e^{\frac{1}{2}\ln(1-t^2)} = (1-t^2)^{\frac{1}{2}}$

Then $\frac{d}{dt}(\mu(t) y(t)) = t(1-t^2)^{1/2}$

$\therefore \mu(t)y(t) - \cancel{\mu(0)}^{1}\cancel{y(0)}^{2} = \int_0^t \tau(1-\tau^2)^{1/2}\, d\tau$

$= -\frac{1}{3}(1-\tau^2)^{3/2}\Big|_0^t$

$= \frac{1}{3}\left(1 - (1-t^2)^{3/2}\right)$

$\therefore y(t) = (1-t^2)^{-\frac{1}{2}}\left[2 + \frac{1}{3} - \frac{1}{3}(1-t^2)^{3/2}\right]$

$$\boxed{y(t) = \frac{7}{3}(1-t^2)^{-\frac{1}{2}} - \frac{1}{3}(1-t^2)}$$

4). $\int (y^4+y^2+2)\,dy = \int \sin(x)\,dx$.

ie: $$\boxed{\frac{y^5}{5} + \frac{y^3}{3} + 2y = -\cos(x) + C_1'}$$

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5) (a) $M = 1 + ye^{xy}$ $\quad N = xe^{xy}$.

$M_y = e^{xy} + xye^{xy}$, $\quad N_x = e^{xy} + yxe^{xy}$.

So $M_y = N_x \Rightarrow$ Equation is exact.

(b) $M = (y^2 + e^{xy})$, $\quad N = xe^{xy}$.

$M_y = 2y + xe^{xy} \neq N_x = e^{xy} + yxe^{xy}$.

Not exact.

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6). $\Psi_x = M = 1 + ye^{xy}$ $\quad$ (1)

$\Psi_y = N = xe^{xy}$ $\quad$ (2)

(2) $\Rightarrow \Psi(x,y) = e^{xy} + C(x)$.

So $\Psi_x = ye^{xy} + C'(x) = 1 + ye^{xy}$ by (1)

$\Rightarrow C'(x) = 1$ $\quad$ ie $C(x) = x + \text{Const}$.

So $$\boxed{\Psi(x,y) = e^{xy} + x + \text{Const}}$$ ← can take to be 0.

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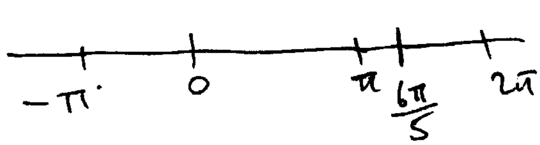
(7) $\sin(t)$ has zeros at $n\pi$



$$\boxed{\text{Solutions exists for } \pi < t < 2\pi}$$

(8) $L(y) = y'' - 2y' + y$.

So $L(e^t) = (1^2 - 2 + 1)\, e^t = 0$

$L(te^t) = (2e^t + te^t) - 2(e^t + te^t) + te^t$

$= 0$ ✓.

Write $Y(t) = c_1 e^{(t-1)} + c_2 (t-1) e^{(t-1)}$

$1 = Y(1) = c_1 + c_2 \cdot 0$ or $c_1 = 1$

$-1 = y'(1) = c_1 + c_2$. $\Rightarrow$ $c_2 = -2$.

So
$$\boxed{\begin{aligned} Y(t) &= e^{(t-1)} - 2(t-1)e^{(t-1)} \\ &= 3e^{(t-1)} - 2te^{(t-1)} \end{aligned}}$$

(9) $W(t) = \begin{vmatrix} e^t \sin(t) & e^{2t} \\ e^t(\sin(t) + \cos(t)) & 2e^{2t} \end{vmatrix} = e^{3t}\left[2\sin(t) - (\sin(t) + \cos(t))\right]$

$$\boxed{W(t) = e^{3t}\left[\sin(t) - \cos(t)\right]}$$