24. Hölder Spaces

Notation 24.1. Let Ω be an open subset of $\mathbb{R}^d$, $BC(\Omega)$ and $BC(\bar{\Omega})$ be the bounded continuous functions on Ω and $\bar{\Omega}$ respectively. By identifying $f \in BC(\bar{\Omega})$ with $f|_\Omega \in BC(\Omega)$, we will consider $BC(\bar{\Omega})$ as a subset of $BC(\Omega)$. For $u \in BC(\Omega)$ and $0 < \beta \leq 1$ let

$$\|u\|_u := \sup_{x\in\Omega} |u(x)| \text{ and } [u]_\beta := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\beta} \right\}.$$

If $[u]_\beta < \infty$, then u is **Hölder continuous** with holder exponent[43] β. The collection of β – Hölder continuous function on Ω will be denoted by

$$C^{0,\beta}(\Omega) := \{u \in BC(\Omega) : [u]_\beta < \infty\}$$

and for $u \in C^{0,\beta}(\Omega)$ let

$$\|u\|_{C^{0,\beta}(\Omega)} := \|u\|_u + [u]_\beta. \tag{24.1}$$

Remark 24.2. If $u : \Omega \to \mathbb{C}$ and $[u]_\beta < \infty$ for some $\beta > 1$, then u is constant on each connected component of Ω. Indeed, if $x \in \Omega$ and $h \in \mathbb{R}^d$ then

$$\left| \frac{u(x+th) - u(x)}{t} \right| \leq [u]_\beta t^\beta / t \to 0 \text{ as } t \to 0$$

which shows $\partial_h u(x) = 0$ for all $x \in \Omega$. If $y \in \Omega$ is in the same connected component as x, then by Exercise 17.5 there exists a smooth curve $\sigma : [0,1] \to \Omega$ such that $\sigma(0) = x$ and $\sigma(1) = y$. So by the fundamental theorem of calculus and the chain rule,

$$u(y) - u(x) = \int_0^1 \frac{d}{dt} u(\sigma(t)) dt = \int_0^1 0 \; dt = 0.$$

This is why we do not talk about Hölder spaces with Hölder exponents larger than 1.

Lemma 24.3. *Suppose $u \in C^1(\Omega) \cap BC(\Omega)$ and $\partial_i u \in BC(\Omega)$ for $i = 1, 2, \ldots, d$, then $u \in C^{0,1}(\Omega)$, i.e. $[u]_1 < \infty$.*

The proof of this lemma is left to the reader as Exercise 24.1.

Theorem 24.4. *Let Ω be an open subset of $\mathbb{R}^d$. Then*

1. *Under the identification of $u \in BC\left(\bar{\Omega}\right)$ with $u|_\Omega \in BC(\Omega)$, $BC(\bar{\Omega})$ is a closed subspace of $BC(\Omega)$.*
2. *Every element $u \in C^{0,\beta}(\Omega)$ has a unique extension to a continuous function (still denoted by u) on $\bar{\Omega}$. Therefore we may identify $C^{0,\beta}(\Omega)$ with $C^{0,\beta}(\bar{\Omega}) \subset BC(\bar{\Omega})$. (In particular we may consider $C^{0,\beta}(\Omega)$ and $C^{0,\beta}(\bar{\Omega})$ to be the same when $\beta > 0$.)*
3. *The function $u \in C^{0,\beta}(\Omega) \to \|u\|_{C^{0,\beta}(\Omega)} \in [0,\infty)$ is a norm on $C^{0,\beta}(\Omega)$ which make $C^{0,\beta}(\Omega)$ into a Banach space.*

Proof. 1. The first item is trivial since for $u \in BC(\bar{\Omega})$, the sup-norm of u on $\bar{\Omega}$ agrees with the sup-norm on Ω and $BC(\bar{\Omega})$ is complete in this norm.

[43] If $\beta = 1$, u is is said to be Lipschitz continuous.

2. Suppose that $[u]_\beta < \infty$ and $x_0 \in \partial\Omega$. Let $\{x_n\}_{n=1}^\infty \subset \Omega$ be a sequence such that $x_0 = \lim_{n\to\infty} x_n$. Then

$$|u(x_n) - u(x_m)| \le [u]_\beta \, |x_n - x_m|^\beta \to 0 \text{ as } m, n \to \infty$$

showing $\{u(x_n)\}_{n=1}^\infty$ is Cauchy so that $\bar{u}(x_0) := \lim_{n\to\infty} u(x_n)$ exists. If $\{y_n\}_{n=1}^\infty \subset \Omega$ is another sequence converging to x_0, then

$$|u(x_n) - u(y_n)| \le [u]_\beta \, |x_n - y_n|^\beta \to 0 \text{ as } n \to \infty,$$

showing $\bar{u}(x_0)$ is well defined. In this way we define $\bar{u}(x)$ for all $x \in \partial\Omega$ and let $\bar{u}(x) = u(x)$ for $x \in \Omega$. Since a similar limiting argument shows

$$|\bar{u}(x) - \bar{u}(y)| \le [u]_\beta \, |x - y|^\beta \text{ for all } x, y \in \bar{\Omega}$$

it follows that $\bar{u}$ is still continuous and $[\bar{u}]_\beta = [u]_\beta$. In the sequel we will abuse notation and simply denote $\bar{u}$ by u.

3. For $u, v \in C^{0,\beta}(\Omega)$,

$$\begin{aligned}[v+u]_\beta &= \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|v(y) + u(y) - v(x) - u(x)|}{|x-y|^\beta} \right\} \\ &\le \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|v(y) - v(x)| + |u(y) - u(x)|}{|x-y|^\beta} \right\} \le [v]_\beta + [u]_\beta\end{aligned}$$

and for $\lambda \in \mathbb{C}$ it is easily seen that $[\lambda u]_\beta = |\lambda| \, [u]_\beta$. This shows $[\cdot]_\beta$ is a semi-norm on $C^{0,\beta}(\Omega)$ and therefore $\|\cdot\|_{C^{0,\beta}(\Omega)}$ defined in Eq. (24.1) is a norm.

To see that $C^{0,\beta}(\Omega)$ is complete, let $\{u_n\}_{n=1}^\infty$ be a $C^{0,\beta}(\Omega)$–Cauchy sequence. Since $BC(\bar{\Omega})$ is complete, there exists $u \in BC(\bar{\Omega})$ such that $\|u - u_n\|_u \to 0$ as $n \to \infty$. For $x, y \in \Omega$ with $x \neq y$,

$$\frac{|u(x) - u(y)|}{|x-y|^\beta} = \lim_{n\to\infty} \frac{|u_n(x) - u_n(y)|}{|x-y|^\beta} \le \limsup_{n\to\infty} [u_n]_\beta \le \lim_{n\to\infty} \|u_n\|_{C^{0,\beta}(\Omega)} < \infty,$$

and so we see that $u \in C^{0,\beta}(\Omega)$. Similarly,

$$\begin{aligned}\frac{|u(x) - u_n(x) - (u(y) - u_n(y))|}{|x-y|^\beta} &= \lim_{m\to\infty} \frac{|(u_m - u_n)(x) - (u_m - u_n)(y)|}{|x-y|^\beta} \\ &\le \limsup_{m\to\infty} [u_m - u_n]_\beta \to 0 \text{ as } n \to \infty,\end{aligned}$$

showing $[u - u_n]_\beta \to 0$ as $n \to \infty$ and therefore $\lim_{n\to\infty} \|u - u_n\|_{C^{0,\beta}(\Omega)} = 0$. ■

Notation 24.5. Since Ω and $\bar{\Omega}$ are locally compact Hausdorff spaces, we may define $C_0(\Omega)$ and $C_0(\bar{\Omega})$ as in Definition 10.29. We will also let

$$C_0^{0,\beta}(\Omega) := C^{0,\beta}(\Omega) \cap C_0(\Omega) \text{ and } C_0^{0,\beta}(\bar{\Omega}) := C^{0,\beta}(\Omega) \cap C_0(\bar{\Omega}).$$

It has already been shown in Proposition 10.30 that $C_0(\Omega)$ and $C_0(\bar{\Omega})$ are closed subspaces of $BC(\Omega)$ and $BC(\bar{\Omega})$ respectively. The next proposition describes the relation between $C_0(\Omega)$ and $C_0(\bar{\Omega})$.

Proposition 24.6. *Each $u \in C_0(\Omega)$ has a unique extension to a continuous function on $\bar{\Omega}$ given by $\bar{u} = u$ on Ω and $\bar{u} = 0$ on $\partial\Omega$ and the extension $\bar{u}$ is in $C_0(\bar{\Omega})$. Conversely if $u \in C_0(\bar{\Omega})$ and $u|_{\partial\Omega} = 0$, then $u|_\Omega \in C_0(\Omega)$. In this way we may identify $C_0(\Omega)$ with those $u \in C_0(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$.*

Proof. Any extension $u \in C_0(\Omega)$ to an element $\bar{u} \in C(\bar{\Omega})$ is necessarily unique, since Ω is dense inside $\bar{\Omega}$. So define $\bar{u} = u$ on Ω and $\bar{u} = 0$ on $\partial\Omega$. We must show $\bar{u}$ is continuous on $\bar{\Omega}$ and $\bar{u} \in C_0(\bar{\Omega})$.

For the continuity assertion it is enough to show $\bar{u}$ is continuous at all points in $\partial\Omega$. For any $\epsilon > 0$, by assumption, the set $K_\epsilon := \{x \in \Omega : |u(x)| \geq \epsilon\}$ is a compact subset of Ω. Since $\partial\Omega = \bar{\Omega} \setminus \Omega$, $\partial\Omega \cap K_\epsilon = \emptyset$ and therefore the distance, $\delta := d(K_\epsilon, \partial\Omega)$, between K_ϵ and $\partial\Omega$ is positive. So if $x \in \partial\Omega$ and $y \in \bar{\Omega}$ and $|y - x| < \delta$, then $|\bar{u}(x) - \bar{u}(y)| = |u(y)| < \epsilon$ which shows $\bar{u} : \bar{\Omega} \to \mathbb{C}$ is continuous. This also shows $\{|\bar{u}| \geq \epsilon\} = \{|u| \geq \epsilon\} = K_\epsilon$ is compact in Ω and hence also in $\bar{\Omega}$. Since $\epsilon > 0$ was arbitrary, this shows $\bar{u} \in C_0(\bar{\Omega})$.

Conversely if $u \in C_0(\bar{\Omega})$ such that $u|_{\partial\Omega} = 0$ and $\epsilon > 0$, then $K_\epsilon := \left\{x \in \bar{\Omega} : |u(x)| \geq \epsilon\right\}$ is a compact subset of $\bar{\Omega}$ which is contained in Ω since $\partial\Omega \cap K_\epsilon = \emptyset$. Therefore K_ϵ is a compact subset of Ω showing $u|_\Omega \in C_0(\bar{\Omega})$. ■

Definition 24.7. Let Ω be an open subset of $\mathbb{R}^d$, $k \in \mathbb{N} \cup \{0\}$ and $\beta \in (0, 1]$. Let $BC^k(\Omega)$ ($BC^k(\bar{\Omega})$) denote the set of $k-$times continuously differentiable functions u on Ω such that $\partial^\alpha u \in BC(\Omega)$ ($\partial^\alpha u \in BC(\bar{\Omega})$)[44] for all $|\alpha| \leq k$. Similarly, let $BC^{k,\beta}(\Omega)$ denote those $u \in BC^k(\Omega)$ such that $[\partial^\alpha u]_\beta < \infty$ for all $|\alpha| = k$. For $u \in BC^k(\Omega)$ let

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u \text{ and}$$

$$\|u\|_{C^{k,\beta}(\overline{\Omega})} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_u + \sum_{|\alpha| = k} [\partial^\alpha u]_\beta.$$

Theorem 24.8. *The spaces $BC^k(\Omega)$ and $BC^{k,\beta}(\Omega)$ equipped with $\|\cdot\|_{C^k(\Omega)}$ and $\|\cdot\|_{C^{k,\beta}(\overline{\Omega})}$ respectively are Banach spaces and $BC^k(\bar{\Omega})$ is a closed subspace of $BC^k(\Omega)$ and $BC^{k,\beta}(\Omega) \subset BC^k(\bar{\Omega})$. Also*

$$C_0^{k,\beta}(\Omega) = C_0^{k,\beta}(\bar{\Omega}) = \{u \in BC^{k,\beta}(\Omega) : \partial^\alpha u \in C_0(\Omega)\ \forall\ |\alpha| \leq k\}$$

is a closed subspace of $BC^{k,\beta}(\Omega)$.

Proof. Suppose that $\{u_n\}_{n=1}^\infty \subset BC^k(\Omega)$ is a Cauchy sequence, then $\{\partial^\alpha u_n\}_{n=1}^\infty$ is a Cauchy sequence in $BC(\Omega)$ for $|\alpha| \leq k$. Since $BC(\Omega)$ is complete, there exists $g_\alpha \in BC(\Omega)$ such that $\lim_{n\to\infty} \|\partial^\alpha u_n - g_\alpha\|_u = 0$ for all $|\alpha| \leq k$. Letting $u := g_0$, we must show $u \in C^k(\Omega)$ and $\partial^\alpha u = g_\alpha$ for all $|\alpha| \leq k$. This will be done by induction on $|\alpha|$. If $|\alpha| = 0$ there is nothing to prove. Suppose that we have verified $u \in C^l(\Omega)$ and $\partial^\alpha u = g_\alpha$ for all $|\alpha| \leq l$ for some $l < k$. Then for $x \in \Omega$, $i \in \{1, 2, \ldots, d\}$ and $t \in \mathbb{R}$ sufficiently small,

$$\partial^a u_n(x + te_i) = \partial^a u_n(x) + \int_0^t \partial_i \partial^a u_n(x + \tau e_i) d\tau.$$

Letting $n \to \infty$ in this equation gives

$$\partial^a u(x + te_i) = \partial^a u(x) + \int_0^t g_{\alpha+e_i}(x + \tau e_i) d\tau$$

from which it follows that $\partial_i \partial^\alpha u(x)$ exists for all $x \in \Omega$ and $\partial_i \partial^\alpha u = g_{\alpha+e_i}$. This completes the induction argument and also the proof that $BC^k(\Omega)$ is complete.

[44] To say $\partial^\alpha u \in BC(\bar{\Omega})$ means that $\partial^\alpha u \in BC(\Omega)$ and $\partial^\alpha u$ extends to a continuous function on $\bar{\Omega}$.

It is easy to check that $BC^k(\bar{\Omega})$ is a closed subspace of $BC^k(\Omega)$ and by using Exercise 24.1 and Theorem 24.4 that that $BC^{k,\beta}(\Omega)$ is a subspace of $BC^k(\bar{\Omega})$. The fact that $C_0^{k,\beta}(\Omega)$ is a closed subspace of $BC^{k,\beta}(\Omega)$ is a consequence of Proposition 10.30.

To prove $BC^{k,\beta}(\Omega)$ is complete, let $\{u_n\}_{n=1}^\infty \subset BC^{k,\beta}(\Omega)$ be a $\|\cdot\|_{C^{k,\beta}(\overline{\Omega})}$ – Cauchy sequence. By the completeness of $BC^k(\Omega)$ just proved, there exists $u \in BC^k(\Omega)$ such that $\lim_{n\to\infty}\|u-u_n\|_{C^k(\Omega)} = 0$. An application of Theorem 24.4 then shows $\lim_{n\to\infty}\|\partial^\alpha u_n - \partial^\alpha u\|_{C^{0,\beta}(\Omega)} = 0$ for $|\alpha| = k$ and therefore $\lim_{n\to\infty}\|u - u_n\|_{C^{k,\beta}(\overline{\Omega})} = 0$. ■

The reader is asked to supply the proof of the following lemma.

Lemma 24.9. *The following inclusions hold. For any* $\beta \in [0,1]$

$$BC^{k+1,0}(\Omega) \subset BC^{k,1}(\Omega) \subset BC^{k,\beta}(\Omega)$$
$$BC^{k+1,0}(\bar{\Omega}) \subset BC^{k,1}(\bar{\Omega}) \subset BC^{k,\beta}(\Omega).$$

Definition 24.10. Let $A : X \to Y$ be a bounded operator between two (separable) Banach spaces. Then A is **compact** if $A\left[B_X(0,1)\right]$ is precompact in Y or equivalently for any $\{x_n\}_{n=1}^\infty \subset X$ such that $\|x_n\| \le 1$ for all n the sequence $y_n := Ax_n \in Y$ has a convergent subsequence.

Example 24.11. Let $X = \ell^2 = Y$ and $\lambda_n \in \mathbb{C}$ such that $\lim_{n\to\infty}\lambda_n = 0$, then $A : X \to Y$ defined by $(Ax)(n) = \lambda_n x(n)$ is compact.

Proof. Suppose $\{x_j\}_{j=1}^\infty \subset \ell^2$ such that $\|x_j\|^2 = \sum |x_j(n)|^2 \le 1$ for all j. By Cantor's Diagonalization argument, there exists $\{j_k\} \subset \{j\}$ such that, for each n, $\tilde{x}_k(n) = x_{j_k}(n)$ converges to some $\tilde{x}(n) \in \mathbb{C}$ as $k \to \infty$. Since for any $M < \infty$,

$$\sum_{n=1}^M |\tilde{x}(n)|^2 = \lim_{k\to\infty}\sum_{n=1}^M |\tilde{x}_k(n)|^2 \le 1$$

we may conclude that $\sum\limits_{n=1}^\infty |\tilde{x}(n)|^2 \le 1$, i.e. $\tilde{x} \in \ell^2$.

Let $y_k := A\tilde{x}_k$ and $y := A\tilde{x}$. We will finish the verification of this example by showing $y_k \to y$ in ℓ^2 as $k \to \infty$. Indeed if $\lambda_M^* = \max\limits_{n\ge M}|\lambda_n|$, then

$$\begin{aligned}
\|A\tilde{x}_k - A\tilde{x}\|^2 &= \sum_{n=1}^\infty |\lambda_n|^2 \left|\tilde{x}_k(n) - \tilde{x}(n)\right|^2 \\
&= \sum_{n=1}^M |\lambda_n|^2|\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \sum_{M+1}^\infty |\tilde{x}_k(n) - \tilde{x}(n)|^2 \\
&\le \sum_{n=1}^M |\lambda_n|^2|\tilde{x}_k(n) - \tilde{x}(n)|^2 + |\lambda_M^*|^2 \left\|\tilde{x}_k - \tilde{x}\right\|^2 \\
&\le \sum_{n=1}^M |\lambda_n|^2|\tilde{x}_k(n) - \tilde{x}(n)|^2 + 4|\lambda_M^*|^2.
\end{aligned}$$

Passing to the limit in this inequality then implies

$$\limsup_{k\to\infty}\|A\tilde{x}_k - A\tilde{x}\|^2 \le 4|\lambda_M^*|^2 \to 0 \text{ as } M \to \infty.$$

■

Lemma 24.12. *If $X \xrightarrow{A} Y \xrightarrow{B} Z$ are continuous operators such the either A or B is compact then the composition $BA : X \to Z$ is also compact.*

Proof. If A is compact and B is bounded, then $BA(B_X(0,1)) \subset B(\overline{AB_X(0,1)})$ which is compact since the image of compact sets under continuous maps are compact. Hence we conclude that $\overline{BA(B_X(0,1))}$ is compact, being the closed subset of the compact set $B(\overline{AB_X(0,1)})$.

If A is continuos and B is compact, then $A(B_X(0,1))$ is a bounded set and so by the compactness of B, $BA(B_X(0,1))$ is a precompact subset of Z, i.e. BA is compact. ■

Proposition 24.13. *Let $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is compact and $0 \le \alpha < \beta \le 1$. Then the inclusion map $i : C^\beta(\overline{\Omega}) \hookrightarrow C^\alpha(\overline{\Omega})$ is compact.*

Let $\{u_n\}_{n=1}^\infty \subset C^\beta(\overline{\Omega})$ such that $\|u_n\|_{C^\beta} \le 1$, i.e. $\|u_n\|_\infty \le 1$ and

$$|u_n(x) - u_n(y)| \le |x-y|^\beta \text{ for all } x, y \in \overline{\Omega}.$$

By Arzela-Ascoli, there exists a subsequence of $\{\tilde{u}_n\}_{n=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ and $u \in C^o(\bar{\Omega})$ such that $\tilde{u}_n \to u$ in C^0. Since

$$|u(x) - u(y)| = \lim_{n\to\infty} |\tilde{u}_n(x) - \tilde{u}_n(y)| \le |x-y|^\beta,$$

$u \in C^\beta$ as well. Define $g_n := u - \tilde{u}_n \in C^\beta$, then

$$[g_n]_\beta + \|g_n\|_{C^0} = \|g_n\|_{C^\beta} \le 2$$

and $g_n \to 0$ in C^0. To finish the proof we must show that $g_n \to 0$ in C^α. Given $\delta > 0$,

$$[g_n]_\alpha = \sup_{x \ne y} \frac{|g_n(x) - g_n(y)|}{|x-y|^\alpha} \le A_n + B_n$$

where

$$\begin{aligned} A_n &= \sup\left\{ \frac{|g_n(x) - g_n(y)|}{|x-y|^\alpha} : x \ne y \text{ and } |x-y| \le \delta \right\} \\ &= \sup\left\{ \frac{|g_n(x) - g_n(y)|}{|x-y|^\beta} \cdot |x-y|^{\beta-\alpha} : x \ne y \text{ and } |x-y| \le \delta \right\} \\ &\le \delta^{\beta-\alpha} \cdot [g_n]_\beta \le 2\delta^{\beta-\alpha} \end{aligned}$$

and

$$B_n = \sup\left\{ \frac{|g_n(x) - g_n(y)|}{|x-y|^\alpha} : |x-y| > \delta \right\} \le 2\delta^{-\alpha} \|g_n\|_{C^0} \to 0 \text{ as } n \to \infty.$$

Therefore,

$$\limsup_{n\to\infty} [g_n]_\alpha \le \limsup_{n\to\infty} A_n + \limsup_{n\to\infty} B_n \le 2\delta^{\beta-\alpha} + 0 \to 0 \text{ as } \delta \downarrow 0.$$

This proposition generalizes to the following theorem which the reader is asked to prove in Exercise 24.2 below.

Theorem 24.14. *Let Ω be a precompact open subset of $\mathbb{R}^d$, $\alpha, \beta \in [0,1]$ and $k, j \in \mathbb{N}_0$. If $j + \beta > k + \alpha$, then $C^{j,\beta}\left(\bar{\Omega}\right)$ is compactly contained in $C^{k,\alpha}\left(\bar{\Omega}\right)$.*

24.1. **Exercises.**

Exercise 24.1. Prove Lemma 24.3.

Exercise 24.2. Prove Theorem 24.14. **Hint:** First prove $C^{j,\beta}\left(\bar{\Omega}\right) \sqsubset\sqsubset C^{j,\alpha}\left(\bar{\Omega}\right)$ is compact if $0 \leq \alpha < \beta \leq 1$. Then use Lemma 24.12 repeatedly to handle all of the other cases.