35. Compact and Fredholm Operators and the Spectral Theorem

In this section H and B will be Hilbert spaces. Typically H and B will be separable, but we will not assume this until it is needed later.

35.1. Compact Operators.

Proposition 35.1. *Let M be a finite dimensional subspace of a Hilbert space H then*

(1) *M is complete (hence closed).*
(2) *Closed bounded subsets of M are compact.*

Proof. Using the Gram-Schmidt procedure, we may choose an orthonormal basis $\{\phi_1, \dots, \phi_n\}$ of M. Define $U : M \to \mathbb{C}^n$ to be the unique unitary map such that $U\phi_i = e_i$ where e_i is the i^{th} standard basis vector in $\mathbb{C}^n$. It now follows that M is complete and that closed bounded subsets of M are compact since the same is true for $\mathbb{C}^n$. ∎

Definition 35.2. A bounded operator $K : H \to B$ is **compact** if K maps bounded sets into precompact sets, i.e. $\overline{K(U)}$ is compact in B, where $U := \{x \in H : \|x\| < 1\}$ is the **unit ball** in H. Equivalently, for all bounded sequences $\{x_n\}_{n=1}^\infty \subset H$, the sequence $\{Kx_n\}_{n=1}^\infty$ has a convergent subsequence in B.

Notice that if $\dim(H) = \infty$ and $T : H \to B$ is invertible, then T is **not** compact.

Definition 35.3. $K : H \to B$ is said to have **finite rank** if $\mathrm{Ran}(K) \subset B$ is finite dimensional.

Corollary 35.4. *If $K : H \to B$ is a finite rank operator, then K is compact. In particular if either $\dim(H) < \infty$ or $\dim(B) < \infty$ then any bounded operator $K : H \to B$ is finite rank and hence compact.*

Example 35.5. Let (X, μ) be a measure space, $H = L^2(X, \mu)$ and

$$k(x, y) \equiv \sum_{i=1}^{n} f_i(x) g_i(y)$$

where

$$f_i, g_i \in L^2(X, \mu) \text{ for } i = 1, \dots, n.$$

Define $(Kf)(x) = \int_X k(x, y) f(y) d\mu(y)$, then $K : L^2(X, \mu) \to L^2(X, \mu)$ is a finite rank operator and hence compact.

Lemma 35.6. *Let $\mathcal{K} := \mathcal{K}(H, B)$ denote the compact operators from H to B. Then $\mathcal{K}(H, B)$ is a norm closed subspace of $L(H, B)$.*

Proof. The fact that $\mathcal{K}$ is a vector subspace of $L(H, B)$ will be left to the reader. Now let $K_n : H \to B$ be compact operators and $K : H \to B$ be a bounded operator such that $\lim_{n\to\infty} \|K_n - K\|_{op} = 0$. We will now show K is compact.

First Proof. Given $\epsilon > 0$, choose $N = N(\epsilon)$ such that $\|K_N - K\| < \epsilon$. Using the fact that $K_N U$ is precompact, choose a finite subset $\Lambda \subset U$ such that $\min_{x\in\Lambda} \|y - K_N x\| < \epsilon$ for all $y \in K_N(U)$. Then for $z = Kx_0 \in K(U)$ and $x \in \Lambda$,

$$\begin{aligned}\|z - Kx\| &= \|(K - K_N)x_0 + K_N(x_0 - x) + (K_N - K)x\| \\ &\le 2\epsilon + \|K_N x_0 - K_N x\|.\end{aligned}$$

Therefore $\min_{x\in\Lambda}\|z-K_N x\|<3\epsilon$, which shows $K(U)$ is 3ϵ bounded for all $\epsilon>0$, $K(U)$ is totally bounded and hence precompact.

Second Proof. Suppose $\{x_n\}_{n=1}^\infty$ is a bounded sequence in H. By compactness, there is a subsequence $\left\{x_n^1\right\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that $\left\{K_1 x_n^1\right\}_{n=1}^\infty$ is convergent in B. Working inductively, we may construct subsequences

$$\{x_n\}_{n=1}^\infty\supset\left\{x_n^1\right\}_{n=1}^\infty\supset\left\{x_n^2\right\}_{n=1}^\infty\cdots\supset\{x_n^m\}_{n=1}^\infty\supset\dots$$

such that $\{K_m x_n^m\}_{n=1}^\infty$ is convergent in B for each m. By the usual Cantor's diagonalization procedure, let $y_n:=x_n^n$, then $\{y_n\}_{n=1}^\infty$ is a subsequence of $\{x_n\}_{n=1}^\infty$ such that $\{K_m y_n\}_{n=1}^\infty$ is convergent for all m. Since

$$\begin{aligned}\|Ky_n-Ky_l\|&\le\|(K-K_m)\,y_n\|+\|K_m(y_n-y_l)\|+\|(K_m-K)\,y_l)\|\\&\le 2\,\|K-K_m\|+\|K_m(y_n-y_l)\|\,,\end{aligned}$$

$$\lim\sup_{n,l\to\infty}\|Ky_n-Ky_l\|\le 2\,\|K-K_m\|\to 0\text{ as }m\to\infty,$$

which shows $\{Ky_n\}_{n=1}^\infty$ is Cauchy and hence convergent. ■

Proposition 35.7. *A bounded operator $K:H\to B$ is compact iff there exists finite rank operators, $K_n:H\to B$, such that $\|K-K_n\|\to 0$ as $n\to\infty$.*

Proof. Since $\overline{K(U)}$ is compact it contains a countable dense subset and from this it follows that $\overline{K\left(H\right)}$ is a separable subspace of B. Let $\{\phi_n\}$ be an orthonormal basis for $\overline{K\left(H\right)}\subset B$ and $P_N y=\sum_{n=1}^N(y,\phi_n)\phi_n$ be the orthogonal projection of y onto $\operatorname{span}\{\phi_n\}_{n=1}^N$. Then $\lim_{N\to\infty}\|P_N y-y\|=0$ for all $y\in K(H)$.

Define $K_n\equiv P_n K$ – a finite rank operator on H. For sake of contradiction suppose that $\limsup_{n\to\infty}\|K-K_n\|=\epsilon>0$, in which case there exists $x_{n_k}\in U$ such that $\|(K-K_{n_k})x_{n_k}\|\ge\epsilon$ for all n_k. Since K is compact, by passing to a subsequence if necessary, we may assume $\{Kx_{n_k}\}_{n_k=1}^\infty$ is convergent in B. Letting $y\equiv\lim_{k\to\infty}Kx_{n_k}$,

$$\begin{aligned}\|(K-K_{n_k})x_{n_k}\|&=\|(1-P_{n_k})Kx_{n_k}\|\le\|(1-P_{n_k})(Kx_{n_k}-y)\|+\|(1-P_{n_k})y\|\\&\le\|Kx_{n_k}-y\|+\|(1-P_{n_k})y\|\to 0\text{ as }k\to\infty.\end{aligned}$$

But this contradicts the assumption that ϵ is positive and hence we must have $\lim_{n\to\infty}\|K-K_n\|=0$, i.e. K is an operator norm limit of finite rank operators. The converse direction follows from Corollary 35.4 and Lemma 35.6. ■

Corollary 35.8. *If K is compact then so is K^*.*

Proof. Let $K_n=P_nK$ be as in the proof of Proposition 35.7, then $K_n^*=K^*P_n$ is still finite rank. Furthermore, using Proposition 12.16,

$$\|K^*-K_n^*\|=\|K-K_n\|\to 0\text{ as }n\to\infty$$

showing K^* is a limit of finite rank operators and hence compact. ■

35.2. **Hilbert Schmidt Operators.**

Proposition 35.9. *Let H and B be a separable Hilbert spaces, $K:H\to B$ be a bounded linear operator, $\{e_n\}_{n=1}^\infty$ and $\{u_m\}_{m=1}^\infty$ be orthonormal basis for H and B respectively. Then:*

(1) $\sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{m=1}^{\infty} \|K^*u_m\|^2$ *allowing for the possibility that the sums are infinite. In particular the* ***Hilbert Schmidt norm*** *of* K,

$$\|K\|_{HS}^2 := \sum_{n=1}^{\infty} \|Ke_n\|^2,$$

is well defined independent of the choice of orthonormal basis $\{e_n\}_{n=1}^{\infty}$. *We say* $K : H \to B$ *is a* ***Hilbert Schmidt operator*** *if* $\|K\|_{HS} < \infty$ *and let* $HS(H,B)$ *denote the space of Hilbert Schmidt operators from* H *to* B.

(2) *For all* $K \in L(H,B)$, $\|K\|_{HS} = \|K^*\|_{HS}$ *and*

$$\|K\|_{HS} \geq \|K\|_{op} := \sup\{\|Kh\| : h \in H \ \ni \ \|h\| = 1\}.$$

(3) *The set* $HS(H,B)$ *is a subspace of* $\mathcal{K}(H,B)$ *and* $\|\cdot\|_{HS}$ *is a norm on* $HS(H,B)$ *for which* $(HS(H,B), \|\cdot\|_{HS})$ *is a Hilbert space. The inner product on* $HS(H,B)$ *is given by*

$$(K_1, K_2)_{HS} = \sum_{n=1}^{\infty} (K_1 e_n, K_2 e_n). \tag{35.1}$$

(4) *Let* $P_N x := \sum_{n=1}^{N} (x, e_n) e_n$ *be orthogonal projection onto* $\operatorname{span}\{e_i : i \leq N\} \subset H$ *and for* $K \in HS(H,B)$, *let* $K_n := KP_n$. *Then*

$$\|K - K_N\|_{op}^2 \leq \|K - K_N\|_{HS}^2 \to 0 \text{ as } N \to \infty,$$

which shows that finite rank operators are dense in $(HS(H,B), \|\cdot\|_{HS})$.

(5) *If* L *is another Hilbert space and* $A : L \to H$ *and* $C : B \to L$ *are bounded operators, then*

$$\|KA\|_{HS} \leq \|K\|_{HS} \|A\|_{op} \text{ and } \|CK\|_{HS} \leq \|K\|_{HS} \|C\|_{op}.$$

Proof. Items 1. and 2. By Parsaval's equality and Fubini's theorem for sums,

$$\sum_{n=1}^{\infty} \|Ke_n\|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |(Ke_n, u_m)|^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |(e_n, K^* u_m)|^2 = \sum_{m=1}^{\infty} \|K^* u_m\|^2.$$

This proves $\|K\|_{HS}$ is well defined independent of basis and that $\|K\|_{HS} = \|K^*\|_{HS}$. For $x \in H \setminus \{0\}$, $x/\|x\|$ may be taken to be the first element in an orthonormal basis for H and hence

$$\left\| K \frac{x}{\|x\|} \right\| \leq \|K\|_{HS}.$$

Multiplying this inequality by $\|x\|$ shows $\|Kx\| \leq \|K\|_{HS} \|x\|$ and hence $\|K\|_{op} \leq \|K\|_{HS}$.

Item 3. For $K_1, K_2 \in L(H,B)$,

$$\begin{aligned}
\|K_1 + K_2\|_{HS} &= \sqrt{\sum_{n=1}^{\infty} \|K_1 e_n + K_2 e_n\|^2} \\
&\leq \sqrt{\sum_{n=1}^{\infty} [\|K_1 e_n\| + \|K_2 e_n\|]^2} = \|\{\|K_1 e_n\| + \|K_2 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} \\
&\leq \|\{\|K_1 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} + \|\{\|K_2 e_n\|\}_{n=1}^{\infty}\|_{\ell_2} = \|K_1\|_{HS} + \|K_2\|_{HS}.
\end{aligned}$$

From this triangle inequality and the homogeneity properties of $\|\cdot\|_{HS}$, we now easily see that $HS(H,B)$ is a subspace of $\mathcal{K}(H,B)$ and $\|\cdot\|_{HS}$ is a norm on $HS(H,B)$. Since

$$\begin{aligned}\sum_{n=1}^{\infty}|(K_1e_n,K_2e_n)| &\le \sum_{n=1}^{\infty}\|K_1e_n\|\,\|K_2e_n\| \\ &\le \sqrt{\sum_{n=1}^{\infty}\|K_1e_n\|^2}\sqrt{\sum_{n=1}^{\infty}\|K_2e_n\|^2} = \|K_1\|_{HS}\,\|K_2\|_{HS}\,,\end{aligned}$$

the sum in Eq. (35.1) is well defined and is easily checked to define an inner product on $HS(H,B)$ such that $\|K\|_{HS}^2 = (K_1,K_2)_{HS}$. To see that $HS(H,B)$ is complete in this inner product suppose $\{K_m\}_{m=1}^{\infty}$ is a $\|\cdot\|_{HS}$ – Cauchy sequence in $HS(H,B)$. Because $L(H,B)$ is complete, there exists $K\in L(H,B)$ such that $\|K_m-K\|_{op}\to 0$ as $m\to\infty$. Since

$$\sum_{n=1}^{N}\|(K-K_m)\,e_n\|^2 = \lim_{l\to\infty}\sum_{n=1}^{N}\|(K_l-K_m)\,e_n\|^2 \le \lim\sup_{l\to\infty}\|K_l-K_m\|_{HS}\,,$$

$$\begin{aligned}\|K_m-K\|_{HS}^2 &= \sum_{n=1}^{\infty}\|(K-K_m)\,e_n\|^2 = \lim_{N\to\infty}\sum_{n=1}^{N}\|(K-K_m)\,e_n\|^2 \\ &\le \lim\sup_{l\to\infty}\|K_l-K_m\|_{HS}\to 0 \text{ as } m\to\infty.\end{aligned}$$

Item 4. Simply observe,

$$\|K-K_N\|_{op}^2 \le \|K-K_N\|_{HS}^2 = \sum_{n>N}\|Ke_n\|^2 \to 0 \text{ as } N\to\infty.$$

Item 5. For $C\in L(B,L)$ and $K\in L(H,B)$ then

$$\|CK\|_{HS}^2 = \sum_{n=1}^{\infty}\|CKe_n\|^2 \le \|C\|_{op}^2\sum_{n=1}^{\infty}\|Ke_n\|^2 = \|C\|_{op}^2\,\|K\|_{HS}^2$$

and for $A\in L\,(L,H)\,,$

$$\|KA\|_{HS} = \|A^*K^*\|_{HS} \le \|A^*\|_{op}\,\|K^*\|_{HS} = \|A\|_{op}\,\|K\|_{HS}\,.$$

■

Remark 35.10. The separability assumptions made in Proposition 35.9 are unnecessary. In general, we define

$$\|K\|_{HS}^2 = \sum_{e\in\Gamma}\|Ke\|^2$$

where $\Gamma\subset H$ is an orthonormal basis. The same proof of Item 1. of Proposition 35.9 shows $\|K\|_{HS}$ is well defined and $\|K\|_{HS} = \|K^*\|_{HS}$. If $\|K\|_{HS}^2<\infty$, then there exists a countable subset $\Gamma_0\subset\Gamma$ such that $Ke=0$ if $e\in\Gamma\setminus\Gamma_0$. Let $H_0 := \overline{\text{span}(\Gamma_0)}$ and $B_0 := \overline{K(H_0)}$. Then $K\,(H)\subset B_0$, $K|_{H_0^\perp}=0$ and hence by applying the results of Proposition 35.9 to $K|_{H_0} : H_0\to B_0$ one easily sees that the separability of H and B are unnecessary in Proposition 35.9.

Exercise 35.1. Suppose that (X, μ) is a σ–finite measure space such that $H = L^2(X, \mu)$ is separable and $k : X \times X \to \mathbb{R}$ is a measurable function, such that

$$\|k\|^2_{L^2(X\times X,\mu\otimes\mu)} \equiv \int_{X\times X} |k(x,y)|^2 d\mu(x)d\mu(y) < \infty.$$

Define, for $f \in H$,

$$Kf(x) = \int_X k(x,y)f(y)d\mu(y),$$

when the integral makes sense. Show:

(1) $Kf(x)$ is defined for μ–a.e. x in X.
(2) The resulting function Kf is in H and $K : H \to H$ is linear.
(3) $\|K\|_{HS} = \|k\|_{L^2(X\times X,\mu\otimes\mu)} < \infty$. (This implies $K \in HS(H,H)$.)

35.1. Since

$$\int_X d\mu(x)\left(\int_X |k(x,y)f(y)|\,d\mu(y)\right)^2 \leq \int_X d\mu(x)\left(\int_X |k(x,y)|^2\,d\mu(y)\right)\left(\int_X |f(y)|^2\,d\mu(y)\right)$$

$$\leq \|k\|_2^2\,\|f\|_2^2 < \infty, \tag{35.2}$$

we learn Kf is almost everywhere defined and that $Kf \in H$. The linearity of K is a consequence of the linearity of the Lebesgue integral. Now suppose $\{\phi_n\}_{n=1}^\infty$ is an orthonormal basis for H. From the estimate in Eq. (35.2), $k(x,\cdot) \in H$ for μ – a.e. $x \in X$ and therefore

$$\begin{aligned}\|K\|^2_{HS} &= \sum_{n=1}^\infty \int_X d\mu(x)\left|\int_X k(x,y)\phi_n(y)d\mu(y)\right|^2 \\ &= \sum_{n=1}^\infty \int_X d\mu(x)\left|(\phi_n, \bar{k}(x,\cdot))\right|^2 = \int_X d\mu(x)\sum_{n=1}^\infty \left|(\phi_n, \bar{k}(x,\cdot))\right|^2 \\ &= \int_X d\mu(x)\left\|\bar{k}(x,\cdot)\right\|^2_H = \int_X d\mu(x)\int_X d\mu(y)\,|k(x,y)|^2 = \|k\|_2^2.\end{aligned}$$

■

Example 35.11. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded set, $\alpha < n$, then the operator $K : L^2(\Omega, m) \to L^2(\Omega, m)$ defined by

$$Kf(x) := \int_\Omega \frac{1}{|x-y|^\alpha} f(y)dy$$

is compact.

Proof. For $\epsilon \geq 0$, let

$$K_\epsilon f(x) := \int_\Omega \frac{1}{|x-y|^\alpha + \epsilon} f(y)dy = \left[g_\epsilon * (1_\Omega f)\right](x)$$

where $g_\epsilon(x) = \frac{1}{|x|^\alpha+\epsilon}1_C(x)$ with $C \subset \mathbb{R}^n$ a sufficiently large ball such that $\Omega - \Omega \subset C$. Since $\alpha < n$, it follows that

$$g_\epsilon \leq g_0 = |\cdot|^{-\alpha}\,1_C \in L^1(\mathbb{R}^n, m).$$

Hence it follows by Proposition 11.12 **??** that

$$\begin{aligned}\|(K - K_\epsilon)\,f\|_{L^2(\Omega)} &\leq \|(g_0 - g_\epsilon) * (1_\Omega f)\|_{L^2(\mathbb{R}^n)} \\ &\leq \|(g_0 - g_\epsilon)\|_{L^1(\mathbb{R}^n)}\,\|1_\Omega f\|_{L^2(\mathbb{R}^n)} = \|(g_0 - g_\epsilon)\|_{L^1(\mathbb{R}^n)}\,\|f\|_{L^2(\not\leq)}\end{aligned}$$

which implies
(35.3)

$$\|K - K_\epsilon\|_{B(L^2(\Omega))} \le \|g_0 - g_\epsilon\|_{L^1(\mathbb{R}^n)} = \int_C \left| \frac{1}{|x|^\alpha + \epsilon} - \frac{1}{|x|^\alpha} \right| dx \to 0 \text{ as } \epsilon \downarrow 0$$

by the dominated convergence theorem. For any $\epsilon > 0$,

$$\int_{\Omega\times\Omega} \left[\frac{1}{|x-y|^\alpha + \epsilon} \right]^2 dxdy < \infty,$$

and hence K_ϵ is Hilbert Schmidt and hence compact. By Eq. (35.3), $K_\epsilon \to K$ as $\epsilon \downarrow 0$ and hence it follows that K is compact as well. ■

35.3. The Spectral Theorem for Self Adjoint Compact Operators.

Lemma 35.12. *Suppose $T : H \to B$ is a bounded operator, then* $\mathrm{Nul}(T^*) = \mathrm{Ran}(T)^\perp$ *and* $\overline{Ran(T)} = \mathrm{Nul}(T^*)^\perp$.

Proof. An element $y \in B$ is in $\mathrm{Nul}(T^*)$ iff $0 = (T^*y, x) = (y, Ax)$ for all $x \in H$ which happens iff $y \in \mathrm{Ran}(T)^\perp$. Because $\overline{\mathrm{Ran}(T)} = \mathrm{Ran}(T)^{\perp\perp}$, $\overline{\mathrm{Ran}(T)} = \mathrm{Nul}(T^*)^\perp$. ■

For the rest of this section, $T \in \mathcal{K}(H) := \mathcal{K}(H,H)$ will be a self-adjoint compact operator or **S.A.C.O.** for short.

Example 35.13 (Model S.A.C.O.)**.** Let $H = \ell_2$ and T be the diagonal matrix

$$T = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where $\lim_{n\to\infty} |\lambda_n| = 0$ and $\lambda_n \in \mathbb{R}$. Then T is a self-adjoint compact operator. (Prove!)

The main theorem of this subsection states that up to unitary equivalence, Example 35.13 is essentially the most general example of an S.A.C.O.

Theorem 35.14. *Suppose $T \in L(H) := L(H,H)$ is a bounded self-adjoint operator, then*

$$\|T\| = \sup_{f\neq 0} \frac{|(f,Tf)|}{\|f\|^2}.$$

Moreover if there exists a non-zero element $g \in H$ such that

$$\frac{|(Tg,g)|}{\|g\|^2} = \|T\|,$$

then g is an eigenvector of T with $Tg = \lambda g$ and $\lambda \in \{\pm\|T\|\}$.

Proof. Let

$$M \equiv \sup_{f\neq 0} \frac{|(f,Tf)|}{\|f\|^2}.$$

We wish to show $M = \|T\|$. Since $|(f,Tf)| \le \|f\|\|Tf\| \le \|T\|\|f\|^2$, we see $M \le \|T\|$.

Conversely let $f, g \in H$ and compute

$$\begin{aligned}(f+g, T(f+g)) &- (f-g, T(f-g)) \\ &= (f, Tg) + (g, Tf) + (f, Tg) + (g, Tf) \\ &= 2[(f, Tg) + (Tg, f)] = 2[(f, Tg) + \overline{(f, Tg)}] \\ &= 4\mathrm{Re}(f, Tg).\end{aligned}$$

Therefore, if $\|f\| = \|g\| = 1$, it follows that

$$|\mathrm{Re}(f, Tg)| \le \frac{M}{4}\left\{\|f+g\|^2 + \|f-g\|^2\right\} = \frac{M}{4}\left\{2\|f\|^2 + 2\|g\|^2\right\} = M.$$

By replacing f be $e^{i\theta} f$ where θ is chosen so that $e^{i\theta}(f, Tg)$ is real, we find

$$|(f, Tg)| \le M \text{ for all } \|f\| = \|g\| = 1.$$

Hence

$$\|T\| = \sup_{\|f\|=\|g\|=1} |(f, Tg)| \le M.$$

If $g \in H \setminus \{0\}$ and $\|T\| = |(Tg, g)|/\|g\|^2$ then, using the Cauchy Schwarz inequality,

$$\|T\| = \frac{|(Tg, g)|}{\|g\|^2} \le \frac{\|Tg\|}{\|g\|} \le \|T\|. \tag{35.4}$$

This implies $|(Tg, g)| = \|Tg\|\|g\|$ and forces equality in the Cauchy Schwarz inequality. So by Theorem 12.2, Tg and g are linearly dependent, i.e. $Tg = \lambda g$ for some $\lambda \in \mathbb{C}$. Substituting this into (35.4) shows that $|\lambda| = \|T\|$. Since T is self-adjoint,

$$\lambda\|g\|^2 = (\lambda g, g) = (Tg, g) = (g, Tg) = (g, \lambda g) = \bar{\lambda}(g, g),$$

which implies that $\lambda \in \mathbb{R}$ and therefore, $\lambda \in \{\pm\|T\|\}$. ■

Theorem 35.15. *Let T be a S.A.C.O., then either $\lambda = \|T\|$ or $\lambda = -\|T\|$ is an eigenvalue of T.*

Proof. Without loss of generality we may assume that T is non-zero since otherwise the result is trivial. By Theorem 35.14, there exists $f_n \in H$ such that $\|f_n\| = 1$ and

$$\frac{|(f_n, Tf_n)|}{\|f_n\|^2} = |(f_n, Tf_n)| \longrightarrow \|T\| \text{ as } n \to \infty. \tag{35.5}$$

By passing to a subsequence if necessary, we may assume that $\lambda := \lim_{n\to\infty}(f_n, Tf_n)$ exists and $\lambda \in \{\pm\|T\|\}$. By passing to a further subsequence if necessary, we may assume, using the compactness of T, that Tf_n is convergent as well. We now compute:

$$\begin{aligned}0 \le \|Tf_n - \lambda f_n\|^2 &= \|Tf_n\|^2 - 2\lambda(Tf_n, f_n) + \lambda^2 \\ &\le \lambda^2 - 2\lambda(Tf_n, f_n) + \lambda^2 \to \lambda^2 - 2\lambda^2 + \lambda^2 = 0 \text{ as } n \to \infty.\end{aligned}$$

Hence

$$Tf_n - \lambda f_n \to 0 \text{ as } n \to \infty \tag{35.6}$$

and therefore

$$f \equiv \lim_{n\to\infty} f_n = \frac{1}{\lambda}\lim_{n\to\infty} Tf_n$$

exists. By the continuity of the inner product, $\|f\| = 1 \neq 0$. By passing to the limit in Eq. (35.6) we find that $Tf = \lambda f$. ■

Lemma 35.16. *Let $T : H \to H$ be a self-adjoint operator and M be a T – invariant subspace of H, i.e. $T(M) \subset M$. Then $M^\perp$ is also a T – invariant subspace, i.e. $T(M^\perp) \subset M^\perp$.*

Proof. Let $x \in M$ and $y \in M^\perp$, then $Tx \in M$ and hence

$$0 = (Tx, y) = (x, Ty) \text{ for all } x \in M.$$

Thus $Ty \in M^\perp$. ■

Theorem 35.17 (Spectral Theorem)**.** *Suppose that $T : H \to H$ is a non-zero S.A.C.O., then*

1. *there exists at least one eigenvalue $\lambda \in \{\pm\|T\|\}$.*
2. *There are at most countable many* ***non-zero*** *eigenvalues, $\{\lambda_n\}_{n=1}^N$, where $N = \infty$ is allowed. (Unless T is finite rank, N will be infinite.)*
3. *The λ_n's (including multiplicities) may be arranged so that $|\lambda_n| \geq |\lambda_{n+1}|$ for all n. If $N = \infty$ then $\lim_{n\to\infty} |\lambda_n| = 0$. (In particular any eigenspace for T with* ***non-zero*** *eigenvalue is finite dimensional.)*
4. *The eigenvectors $\{\phi_n\}_{n=1}^N$ can be chosen to be an O.N. set such that $H = \overline{span\{\phi_n\}} \oplus \mathrm{Nul}(T)$.*
5. *Using the $\{\phi_n\}_{n=1}^N$ above,*
$$T\psi = \sum_{n=1}^N \lambda_n(\psi, \phi_n)\phi_n \text{ for all } \psi \in H.$$
6. *The spectrum of T is $\sigma(T) = \{0\} \cup \cup_{n=1}^\infty \{\lambda_n\}$.*

Proof. We will find λ_n's and ϕ_n's recursively. Let $\lambda_1 \in \{\pm\|T\|\}$ and $\phi_1 \in H$ such that $T\phi_1 = \lambda_1\phi_1$ as in Theorem 35.15. Take $M_1 = \mathrm{span}(\phi_1)$ so $T(M_1) \subset M_1$. By Lemma 35.16, $TM_1^\perp \subset M_1^\perp$. Define $T_1 : M_1^\perp \to M_1^\perp$ via $T_1 = T|_{M_1^\perp}$. Then T_1 is again a compact operator. If $T_1 = 0$, we are done.

If $T_1 \neq 0$, by Theorem 35.15 there exists $\lambda_2 \in \{\pm\|T\|_1\}$ and $\phi_2 \in M_1^\perp$ such that $\|\phi_2\| = 1$ and $T_1\phi_2 = T\phi_2 = \lambda_2\phi_2$. Let $M_2 \equiv \mathrm{span}(\phi_1, \phi_2)$. Again $T(M_2) \subset M_2$ and hence $T_2 \equiv T|_{M_2^\perp} : M_2^\perp \to M_2^\perp$ is compact. Again if $T_2 = 0$ we are done.

If $T_2 \neq 0$. Then by Theorem 35.15 there exists $\lambda_3 \in \{\pm\|T\|_2\}$ and $\phi_3 \in M_2^\perp$ such that $\|\phi_3\| = 1$ and $T_2\phi_3 = T\phi_3 = \lambda_3\phi_3$. Continuing this way indefinitely or until we reach a point where $T_n = 0$, we construct a sequence $\{\lambda_n\}_{n=1}^N$ of eigenvalues and orthonormal eigenvectors $\{\phi_n\}_{n=1}^N$ such that $|\lambda_i| \geq |\lambda_{i+1}|$ with the further property that

$$|\lambda_i| = \sup_{\phi\perp\{\phi_1,\phi_2,\dots\phi_{i-1}\}} \frac{\|T\phi\|}{\|\phi\|} \tag{35.7}$$

If $N = \infty$ then $\lim_{i\to\infty} |\lambda_i| = 0$ for if not there would exist $\epsilon > 0$ such that $|\lambda_i| \geq \epsilon > 0$ for all i. In this case $\{\phi_i/\lambda_i\}_{i=1}^\infty$ is sequence in H bounded by ϵ^{-1}. By compactness of T, there exists a subsequence i_k such that $\phi_{i_k} = T\phi_{i_k}/\lambda_{i_k}$ is convergent. But this is impossible since $\{\phi_{i_k}\}$ is an orthonormal set. Hence we must have that $\epsilon = 0$.

Let $M \equiv \text{span}\{\phi_i\}_{i=1}^N$ with $N = \infty$ **possible**. Then $T(M) \subset M$ and hence $T(M^\perp) \subset M^\perp$. Using Eq. (35.7),

$$\|T|_{M^\perp}\| \le \|T|_{M_n^\perp}\| = |\lambda_n| \longrightarrow 0 \text{ as } n \to \infty$$

showing $T|M^\perp \equiv 0$.

Define P_0 to be orthogonal projection onto $M^\perp$. Then for $\psi \in H$,

$$\psi = P_0\psi + (1 - P_0)\psi = P_0\psi + \sum_{i=1}^N (\psi, \phi_i)\phi_i$$

and

$$T\psi = TP_0\psi + T\sum_{i=1}^N (\psi, \phi_i)\phi_i = \sum_{i=1}^N \lambda_i(\psi, \phi_i)\phi_i.$$

Since $\{\lambda_n\} \subset \sigma(T)$ and $\sigma(T)$ is closed, it follows that $0 \in \sigma(T)$ and hence $\{\lambda_n\}_{n=1}^\infty \cup \{0\} \subset \sigma(T)$. Suppose that $z \notin \{\lambda_n\}_{n=1}^\infty \cup \{0\}$ and let d be the distance between z and $\{\lambda_n\}_{n=1}^\infty \cup \{0\}$. Notice that $d > 0$ because $\lim_{n\to\infty} \lambda_n = 0$. A few simple computations show that:

$$(T - zI)\psi = \sum_{i=1}^N (\psi, \phi_i)(\lambda_i - z)\phi_i - zP_0\psi,$$

$(T - z)^{-1}$ exists,

$$(T - zI)^{-1}\psi = \sum_{i=1}^N (\psi, \phi_i)(\lambda_i - z)^{-1}\phi_i - z^{-1}P_0\psi,$$

and

$$\begin{aligned}\|(T - zI)^{-1}\psi\|^2 &= \sum_{i=1}^N |(\psi, \phi_i)|^2 \frac{1}{|\lambda_i - z|^2} + \frac{1}{|z|^2}\|P_0\psi\|^2 \\ &\le \left(\frac{1}{d}\right)^2 \left(\sum_{i=1}^N |(\psi, \phi_i)|^2 + \|P_0\psi\|^2\right) = \frac{1}{d^2}\|\psi\|^2.\end{aligned}$$

We have thus shown that $(T - zI)^{-1}$ exists, $\|(T - zI)^{-1}\| \le d^{-1} < \infty$ and hence $z \notin \sigma(T)$. ■

35.4. Structure of Compact Operators.

Theorem 35.18. *Let $K : H \to B$ be a compact operator. Then there exists $N \in \mathbb{N} \cup \{\infty\}$, orthonormal subsets $\{\phi_n\}_{n=1}^N \subset H$ and $\{\psi_n\}_{n=1}^N \subset B$ and a sequences $\{\lambda_n\}_{n=1}^N \subset \mathbb{C}$ such that $\lim_{n\to\infty} \lambda_n = 0$ if $N = \infty$ and*

$$Kf = \sum_{n=1}^N \lambda_n(f, \phi_n)\psi_n \text{ for all } f \in H.$$

Proof. The operator $K^*K \in \mathcal{K}(H)$ is self-adjoint and hence by Theorem 35.17, there exists an orthonormal set $\{\phi_n\}_{n=1}^N \subset H$ and $\{\mu_n\}_{n=1}^\infty \subset (0, \infty)$ such that

$$K^*Kf = \sum_{n=1}^N \mu_n(f, \phi_n)\phi_n \text{ for all } f \in H.$$

Let $\lambda_n := \sqrt{\mu_n}$ and $\sqrt{K^*K} \in \mathcal{K}(H)$ be defined by

$$\sqrt{K^*K} f = \sum_{n=1}^N \lambda_n (f, \phi_n) \phi_n \text{ for all } f \in H.$$

Define $U \in L(H, B)$ so that $U =$ "$K(K^*K)^{-1/2}$," or more precisely by

$$Uf = \sum_{n=1}^N \lambda_n^{-1} (f, \phi_n) K\phi_n. \tag{35.8}$$

The operator U is well defined because

$$(\lambda_n^{-1} K\phi_n, \lambda_m^{-1} K\phi_m) = \lambda_n^{-1}\lambda_m^{-1}(\phi_n, K^*K\phi_m) = \lambda_n^{-1}\lambda_m^{-1}\lambda_m^2 \delta_{m,n} = \delta_{m,n}$$

which shows $\left\{\lambda_n^{-1} K\phi_n\right\}_{n=1}^\infty$ is an orthonormal subset of B. Moreover this also shows

$$\|Uf\|^2 = \sum_{n=1}^N |(f, \phi_n)|^2 = \|Pf\|^2$$

where $P = P_{\mathrm{Nul}(K)^\perp}$. Replacing f by $(K^*K)^{1/2} f$ in Eq. (35.8) shows

$$U(K^*K)^{1/2} f = \sum_{n=1}^N \lambda_n^{-1}((K^*K)^{1/2} f, \phi_n) K\phi_n = \sum_{n=1}^N (f, \phi_n) K\phi_n = Kf, \tag{35.9}$$

since $f = \sum_{n=1}^N (f, \phi_n)\phi_n + Pf$.

From Eq. (35.9) it follows that

$$Kf = \sum_{n=1}^N \lambda_n (f, \phi_n) U\phi_n = \sum_{n=1}^N \lambda_n (f, \phi_n) \psi_n$$

where $\{\psi_n\}_{n=1}^N$ is the orthonormal sequence in B defined by

$$\psi_n := U\phi_n = \lambda_n^{-1} K\phi_n.$$

■

35.4.1. *Trace Class Operators.* We will say $K \in \mathcal{K}(H)$ is **trace class** if

$$\mathrm{tr}(\sqrt{K^*K}) := \sum_{n=1}^N \lambda_n < \infty$$

in which case we define

$$\mathrm{tr}(K) = \sum_{n=1}^N \lambda_n (\psi_n, \phi_n).$$

Notice that if $\{e_m\}_{m=1}^\infty$ is any orthonormal basis in H (or for the $\overline{\mathrm{Ran}(K)}$ if H is not separable) then

$$\begin{aligned}\sum_{m=1}^M (Ke_m, e_m) &= \sum_{m=1}^M (\sum_{n=1}^N \lambda_n (e_m, \phi_n)\psi_n, e_m) = \sum_{n=1}^N \lambda_n \sum_{m=1}^M (e_m, \phi_n)(\psi_n, e_m) \\ &= \sum_{n=1}^N \lambda_n (P_M \psi_n, \phi_n)\end{aligned}$$

where P_M is orthogonal projection onto $\mathrm{Span}(e_1, \dots, e_M)$. Therefore by dominated convergence theorem ,

$$\sum_{m=1}^{\infty}(Ke_m, e_m) = \lim_{M\to\infty}\sum_{n=1}^{N}\lambda_n(P_M\psi_n, \phi_n) = \sum_{n=1}^{N}\lambda_n \lim_{M\to\infty}(P_M\psi_n, \phi_n)$$
$$= \sum_{n=1}^{N}\lambda_n(\psi_n, \phi_n) = \mathrm{tr}(K).$$

35.5. **Fredholm Operators.**

Lemma 35.19. *Let $M \subset H$ be a closed subspace and $V \subset H$ be a finite dimensional subspace. Then $M+V$ is closed as well. In particular if $\mathrm{codim}(M) \equiv \dim(H/M) < \infty$ and $W \subset H$ is a subspace such that $M \subset W$, then W is closed and $\mathrm{codim}(W) < \infty$.*

Proof. Let $P : H \to M$ be orthogonal projection and let $V_0 := (I - P)\,V$. Since $\dim(V_0) \le \dim(V) < \infty$, V_0 is still closed. Also it is easily seen that $M+V = M\overset{\perp}{\oplus}V_0$ from which it follows that $M + V$ is closed because $\{z_n = m_n + v_n\} \subset M \overset{\perp}{\oplus} V_0$ is convergent iff $\{m_n\} \subset M$ and $\{v_n\} \subset V_0$ are convergent.

If $\mathrm{codim}(M) < \infty$ and $M \subset W$, there is a finite dimensional subspace $V \subset H$ such that $W = M + V$ and so by what we have just proved, W is closed as well. It should also be clear that $\mathrm{codim}(W) \le \mathrm{codim}(M) < \infty$. ∎

Lemma 35.20. *If $K : H \to B$ is a finite rank operator, then there exists $\{\phi_n\}_{n=1}^k \subset H$ and $\{\psi_n\}_{n=1}^k \subset B$ such that*

1. $Kx = \sum_{n=1}^k (x, \phi_n)\psi_n$ *for all* $x \in H$.
2. $K^* y = \sum_{n=1}^k (y, \psi_n)\phi_n$ *for all* $y \in B$, *in particular* K^* *is still finite rank.*
   For the next two items, further assume $B = H$.
3. $\dim \mathrm{Nul}(I + K) < \infty$.
4. $\dim$ *coker*$(I + K) < \infty$, *Ran*$(I + K)$ *is closed and*
$$\mathrm{Ran}(I + K) = \mathrm{Nul}(I + K^*)^\perp.$$

Proof.

1. Choose $\{\psi_n\}_1^k$ to be an orthonormal basis for $\mathrm{Ran}(K)$. Then for $x \in H$,
$$Kx = \sum_{n=1}^k (Kx, \psi_n)\psi_n = \sum_{n=1}^k (x, K^*\psi_n)\psi_n = \sum_{n=1}^k (x, \phi_n)\psi_n$$
   where $\phi_n \equiv K^*\psi_n$.
2. Item 2. is a simple computation left to the reader.
3. Since $\mathrm{Nul}(I+K) = \{x \in H \mid x = -Kx\} \subset \mathrm{Ran}(K)$ it is finite dimensional.
4. Since $x = (I + K)x \in \mathrm{Ran}(I + K)$for $x \in \mathrm{Nul}(K)$, $\mathrm{Nul}(K) \subset \mathrm{Ran}(I + K)$. Since $\{\phi_1, \phi_2, \dots, \phi_k\}^\perp \subset \mathrm{Nul}(K)$, $H = \mathrm{Nul}(K) + \mathrm{span}\left(\{\phi_1, \phi_2, \dots, \phi_k\}\right)$ and thus $\mathrm{codim}\left(\mathrm{Nul}(K)\right) < \infty$. From these comments and Lemma 35.19, $\mathrm{Ran}(I + K)$ is closed and $\mathrm{codim}\left(\mathrm{Ran}(I + K)\right) \le \mathrm{codim}\left(\mathrm{Nul}(K)\right) < \infty$. The assertion that $\mathrm{Ran}(I+K) = \mathrm{Nul}(I+K^*)^\perp$ is a consequence of Lemma 35.12 below.

∎

Definition 35.21. A bounded operator $F : H \to B$ is **Fredholm** iff the $\dim \operatorname{Nul}(F) < \infty$, $\dim \operatorname{coker}(F) < \infty$ and $\operatorname{Ran}(F)$ is closed in B. (Recall: $\operatorname{coker}(F) \equiv B/\operatorname{Ran}(F)$.) The **index** of F is the integer,

$$\operatorname{index}(F) = \dim \operatorname{Nul}(F) - \dim \operatorname{coker}(F) \tag{35.10}$$

$$= \dim \operatorname{Nul}(F) - \dim \operatorname{Nul}(F^*) \tag{35.11}$$

Notice that equations (35.10) and (35.11) are the same since, (using $\operatorname{Ran}(F)$ is closed)

$$B = \operatorname{Ran}(F) \oplus \operatorname{Ran}(F)^\perp = \operatorname{Ran}(F) \oplus \operatorname{Nul}(F^*)$$

so that $\operatorname{coker}(F) = B/\operatorname{Ran}(F) \cong \operatorname{Nul}(F^*)$.

Lemma 35.22. *The requirement that* $\operatorname{Ran}(F)$ *is closed in Defintion 35.21 is redundant.*

Proof. By restricting F to $\operatorname{Nul}(F)^\perp$, we may assume without loss of generality that $\operatorname{Nul}(F) = \{0\}$. Assuming $\dim \operatorname{coker}(F) < \infty$, there exists a finite dimensional subspace $V \subset B$ such that $B = \operatorname{Ran}(F) \oplus V$. Since V is finite dimensional, V is closed and hence $B = V \oplus V^\perp$. Let $\pi : B \to V^\perp$ be the orthogonal projection operator onto $V^\perp$ and let $G \equiv \pi F : H \to V^\perp$ which is continuous, being the composition of two bounded transformations. Since G is a linear isomorphism, as the reader should check, the open mapping theorem implies the inverse operator $G^{-1} : V^\perp \to H$ is bounded.

Suppose that $h_n \in H$ is a sequence such that $\lim_{n\to\infty} F(h_n) =: b$ exists in B. Then by composing this last equation with π, we find that $\lim_{n\to\infty} G(h_n) = \pi(b)$ exists in $V^\perp$. Composing this equation with G^{-1} shows that $h := \lim_{n\to\infty} h_n = G^{-1}\pi(b)$ exists in H. Therefore, $F(h_n) \to F(h) \in \operatorname{Ran}(F)$, which shows that $\operatorname{Ran}(F)$ is closed. ■

Remark 35.23. It is essential that the subspace $M \equiv \operatorname{Ran}(F)$ in Lemma 35.22 is the image of a bounded operator, for it is not true that every finite codimensional subspace M of a Banach space B is necessarily closed. To see this suppose that B is a separable infinite dimensional Banach space and let $A \subset B$ be an **algebraic** basis for B, which exists by a Zorn's lemma argument. Since $\dim(B) = \infty$ and B is complete, A must be uncountable. Indeed, if A were countable we could write $B = \cup_{n=1}^\infty B_n$ where B_n are finite dimensional (necessarily closed) subspaces of B. This shows that B is the countable union of nowhere dense closed subsets which violates the Baire Category theorem.

By separability of B, there exists a countable subset $A_0 \subset A$ such that the closure of $M_0 \equiv \operatorname{span}(A_0)$ is equal to B. Choose $x_0 \in A \setminus A_0$, and let $M \equiv \operatorname{span}(A \setminus \{x_0\})$. Then $M_{\mathbf{0}} \subset M$ so that $B = \bar{M}_0 = \bar{M}$, while $\operatorname{codim}(M) = 1$. Clearly this M can not be closed.

Example 35.24. Suppose that H and B are finite dimensional Hilbert spaces and $F : H \to B$ is Fredholm. Then

$$\operatorname{index}(F) = \dim(B) - \dim(H). \tag{35.12}$$

The formula in Eq. (35.12) may be verified using the rank nullity theorem,

$$\dim(H) = \dim \operatorname{Nul}(F) + \dim \operatorname{Ran}(F),$$

and the fact that

$$\dim(B/\operatorname{Ran}(F)) = \dim(B) - \dim \operatorname{Ran}(F).$$

Theorem 35.25. *A bounded operator $F : H \to B$ is Fredholm iff there exists a bounded operator $A : B \to H$ such that $AF - I$ and $FA - I$ are both compact operators. (In fact we may choose A so that $AF - I$ and $FA - I$ are both finite rank operators.)*

Proof. (⇒) Suppose F is Fredholm, then $F : \mathrm{Nul}(F)^\perp \to \mathrm{Ran}(F)$ is a bijective bounded linear map between Hilbert spaces. (Recall that $\mathrm{Ran}(F)$ is a closed subspace of B and hence a Hilbert space.) Let $\tilde{F}$ be the inverse of this map—a bounded map by the open mapping theorem. Let $P : H \to \mathrm{Ran}(F)$ be orthogonal projection and set $A \equiv \tilde{F}P$. Then $AF - I = \tilde{F}PF - I = \tilde{F}F - I = -Q$ where Q is the orthogonal projection onto $\mathrm{Nul}(F)$. Similarly, $FA - I = F\tilde{F}P - I = -(I - P)$. Because $I - P$ and Q are finite rank projections and hence compact, both $AF - I$ and $FA - I$ are compact.

(⇐) We first show that the operator $A : B \to H$ may be modified so that $AF - I$ and $FA - I$ are both finite rank operators. To this end let $G \equiv AF - I$ (G is compact) and choose a finite rank approximation G_1 to G such that $G = G_1 + \mathcal{E}$ where $\|\mathcal{E}\| < 1$. Define $A_L : B \to H$ to be the operator $A_L \equiv (I + \mathcal{E})^{-1}A$. Since $AF = (I + \mathcal{E}) + G_1$,

$$A_L F = (I + \mathcal{E})^{-1} AF = I + (I + \mathcal{E})^{-1} G_1 = I + K_L$$

where K_L is a finite rank operator. Similarly there exists a bounded operator $A_R : B \to H$ and a finite rank operator K_R such that $FA_R = I + K_R$. Notice that $A_L F A_R = A_R + K_L A_R$ on one hand and $A_L F A_R = A_L + A_L K_R$ on the other. Therefore, $A_L - A_R = A_L K_R - K_L A_R =: S$ is a finite rank operator. Therefore $FA_L = F(A_R + S) = I + K_R + FS$, so that $FA_L - I = K_R - FS$ is still a finite rank operator. Thus we have shown that there exists a bounded operator $\tilde{A} : B \to H$ such that $\tilde{A}F - I$ and $F\tilde{A} - I$ are both finite rank operators.

We now assume that A is chosen such that $AF - I = G_1$, $FA - I = G_2$ are finite rank. Clearly $\mathrm{Nul}(F) \subset \mathrm{Nul}(AF) = \mathrm{Nul}(I + G_1)$ and $\mathrm{Ran}(F) \supseteq \mathrm{Ran}(FA) = \mathrm{Ran}(I + G_2)$. The theorem now follows from Lemma 35.19 and Lemma 35.20. ■

Corollary 35.26. *If $F : H \longrightarrow B$ is Fredholm then F^* is Fredholm and index$(F) = -$index(F^*).*

Proof. Choose $A : B \to H$ such that both $AF - I$ and $FA - I$ are compact. Then $F^*A^* - I$ and $A^*F^* - I$ are compact which implies that F^* is Fredholm. The assertion, $\mathrm{index}(F) = -\mathrm{index}(F^*)$, follows directly from Eq. (35.11). ■

Lemma 35.27. *A bounded operator $F : H \to B$ is Fredholm if and only if there exists orthogonal decompositions $H = H_1 \oplus H_2$ and $B = B_1 \oplus B_2$ such that*

1. *H_1 and B_1 are closed subspaces,*
2. *H_2 and B_2 are finite dimensional subspaces, and*
3. *F has the block diagonal form*

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix} \longrightarrow \begin{matrix} B_1 \\ \oplus \\ B_2 \end{matrix} \tag{35.13}$$

with $F_{11} : H_1 \to B_1$ being a bounded invertible operator.

Furthermore, given this decomposition, index$(F) = \dim(H_2) - \dim(B_2)$.

Proof. If F is Fredholm, set $H_1 = \mathrm{Nul}(F)^\perp, H_2 = \mathrm{Nul}(F), B_1 = \mathrm{Ran}(F)$, and $B_2 = \mathrm{Ran}(F)^\perp$. Then $F = \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix}$, where $F_{11} \equiv F|_{H_1} : H_1 \to B_1$ is invertible.

For the converse, assume that F is given as in Eq. (35.13). Let $A \equiv \begin{pmatrix} F_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$ then

$$AF = \begin{pmatrix} I & F_{11}^{-1}F_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} 0 & F_{11}^{-1}F_{12} \\ 0 & -I \end{pmatrix},$$

so that $AF - I$ is finite rank. Similarly one shows that $FA - I$ is finite rank, which shows that F is Fredholm.

Now to compute the index of F, notice that $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Nul}(F)$ iff

$$\begin{aligned} F_{11}x_1 + F_{12}x_2 = 0 \\ F_{21}x_1 + F_{22}x_2 = 0 \end{aligned}$$

which happens iff $x_1 = -F_{11}^{-1}F_{12}x_2$ and $(-F_{21}F_{11}^{-1}F_{12} + F_{22})x_2 = 0$. Let $D \equiv (F_{22} - F_{21}F_{11}^{-1}F_{12}) : H_2 \to B_2$, then the mapping

$$x_2 \in \mathrm{Nul}(D) \to \begin{pmatrix} -F_{11}^{-1}F_{12}x_2 \\ x_2 \end{pmatrix} \in \mathrm{Nul}(F)$$

is a linear isomorphism of vector spaces so that $\mathrm{Nul}(F) \cong \mathrm{Nul}(D)$. Since

$$F^* = \begin{pmatrix} F_{11}^* & F_{21}^* \\ F_{12}^* & F_{22}^* \end{pmatrix} \quad \begin{matrix} B_1 \\ \oplus \\ B_2 \end{matrix} \longrightarrow \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix},$$

similar reasoning implies $\mathrm{Nul}(F^*) \cong \mathrm{Nul}(D^*)$. This shows that $\mathrm{index}(F) = \mathrm{index}(D)$. But we have already seen in Example 35.24 that $\mathrm{index}(D) = \dim H_2 - \dim B_2$. ■

Proposition 35.28. *Let F be a Fredholm operator and K be a compact operator from $H \to B$. Further assume $T : B \to X$ (where X is another Hilbert space) is also Fredholm. Then*

1. *the Fredholm operators form an open subset of the bounded operators. Moreover if $\mathcal{E} : H \to B$ is a bounded operator with $\|\mathcal{E}\|$ sufficiently small we have index(F) =index$(F + \mathcal{E})$.*
2. *$F + K$ is Fredholm and index(F) = index$(F + K)$.*
3. *TF is Fredholm and index(TF) = index(T) + index(F)*

Proof.

1. We know F may be written in the block form given in Eq. (35.13) with $F_{11} : H_1 \to B_1$ being a bounded invertible operator. Decompose $\mathcal{E}$ into the block form as
$$\mathcal{E} = \begin{pmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{21} & \mathcal{E}_{22} \end{pmatrix}$$
and choose $\|\mathcal{E}\|$ sufficiently small such that $\|\mathcal{E}_{11}\|$ is sufficiently small to guarantee that $F_{11} + \mathcal{E}_{11}$ is still invertible. (Recall that the invertible operators form an open set.) Thus $F + \mathcal{E} = \begin{pmatrix} F_{11} + \mathcal{E}_{11} & * \\ * & * \end{pmatrix}$ has the block

form of a Fredholm operator and the index may be computed as:

$$\operatorname{index}(F+\mathcal{E}) = \dim H_2 - \dim B_2 = \operatorname{index}(F).$$

(2) Given $K : H \to B$ compact, it is easily seen that $F + K$ is still Fredholm. Indeed if $A : B \to H$ is a bounded operator such that $G_1 \equiv AF - I$ and $G_2 \equiv FA - I$ are both compact, then $A(F + K) - I = G_1 + AK$ and $(F + K)A - I = G_2 + KA$ are both compact. Hence $F + K$ is Fredholm by Theorem 35.25. By item 1., the function $f(t) \equiv \operatorname{index}(F + tK)$ is a continuous locally constant function of $t \in \mathbb{R}$ and hence is constant. In particular, $\operatorname{index}(F + K) = f(1) = f(0) = \operatorname{index}(F)$.

(3) It is easily seen, using Theorem 35.25 that the product of two Fredholm operators is again Fredholm. So it only remains to verify the index formula in item 3.

For this let $H_1 \equiv \operatorname{Nul}(F)^\perp$, $H_2 \equiv \operatorname{Nul}(F)$, $B_1 \equiv \operatorname{Ran}(T) = T(H_1)$, and $B_2 \equiv \operatorname{Ran}(T)^\perp = \operatorname{Nul}(T^*)$. Then F decomposes into the block form:

$$F = \begin{pmatrix} \tilde{F} & 0 \\ 0 & 0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix} \longrightarrow \begin{matrix} B_1 \\ \oplus \\ B_2 \end{matrix},$$

where $\tilde{F} = F|_{H_1} : H_1 \to B_1$ is an invertible operator. Let $Y_1 \equiv T(B_1)$ and $Y_2 \equiv Y_1^\perp = T(B_1)^\perp$. Notice that $Y_1 = T(B_1) = TQ(B_1)$, where $Q : B \to B_1 \subset B$ is orthogonal projection onto B_1. Since B_1 is closed and B_2 is finite dimensional, Q is Fredholm. Hence TQ is Fredholm and $Y_1 = TQ(B_1)$ is closed in Y and is of finite codimension. Using the above decompositions, we may write T in the block form:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : \begin{matrix} B_1 \\ \oplus \\ B_2 \end{matrix} \longrightarrow \begin{matrix} Y_1 \\ \oplus \\ Y_2 \end{matrix}.$$

Since $R = \begin{pmatrix} 0 & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : B \to Y$ is a finite rank operator and hence $RF : H \to Y$ is finite rank, $\operatorname{index}(T-R) = \operatorname{index}(T)$ and $\operatorname{index}(TF-RF) = \operatorname{index}(TF)$. Hence without loss of generality we may assume that T has the form $T = \begin{pmatrix} \tilde{T} & 0 \\ 0 & 0 \end{pmatrix}$, ($\tilde{T} = T_{11}$) and hence

$$TF = \begin{pmatrix} \tilde{T}\tilde{F} & 0 \\ 0 & 0 \end{pmatrix} : \begin{matrix} H_1 \\ \oplus \\ H_2 \end{matrix} \longrightarrow \begin{matrix} Y_1 \\ \oplus \\ Y_2 \end{matrix}.$$

We now compute the $\operatorname{index}(T)$. Notice that $\operatorname{Nul}(T) = \operatorname{Nul}(\tilde{T}) \oplus B_2$ and $\operatorname{Ran}(T) = \tilde{T}(B_1) = Y_1$. So

$$\operatorname{index}(T) = \operatorname{index}(\tilde{T}) + \dim(B_2) - \dim(Y_2).$$

Similarly,

$$\operatorname{index}(TF) = \operatorname{index}(\tilde{T}\tilde{F}) + \dim(H_2) - \dim(Y_2),$$

and as we have already seen

$$\operatorname{index}(F) = \dim(H_2) - \dim(B_2).$$

Therefore,

$$\text{index}(TF) - \text{index}(T) - \text{index}(F) = \text{index}(\tilde{T}\tilde{F}) - \text{index}(\tilde{T}).$$

Since $\tilde{F}$ is invertible, $\text{Ran}(\tilde{T}) = \text{Ran}(\tilde{T}\tilde{F})$ and $\text{Nul}(\tilde{T}) \cong \text{Nul}(\tilde{T}\tilde{F})$. Thus $\text{index}(\tilde{T}\tilde{F}) - \text{index}(\tilde{T}) = 0$ and the theorem is proved.

■

35.6. **Tensor Product Spaces .** References for this section are Reed and Simon [**?**] (Volume 1, Chapter VI.5), Simon [**?**], and Schatten [**?**]. See also Reed and Simon [**?**] (Volume 2 § IX.4 and §XIII.17).

Let H and K be separable Hilbert spaces and $H \otimes K$ will denote the usual Hilbert completion of the algebraic tensors $H \otimes_f K$. Recall that the inner product on $H \otimes K$ is determined by $(h \otimes k, h' \otimes k') = (h, h')(k, k')$. The following proposition is well known.

Proposition 35.29 (Structure of $H \otimes K$)**.** *There is a bounded linear map $T : H \otimes K \to B(K, H)$ determined by*

$$T(h \otimes k)k' \equiv (k, k')h \text{ for all } k, k' \in K \text{ and } h \in H.$$

Moreover $T(H \otimes K) = HS(K, H)$ — the Hilbert Schmidt operators from K to H. The map $T : H \otimes K \to HS(K, H)$ is unitary equivalence of Hilbert spaces. Finally, any $A \in H \otimes K$ may be expressed as

$$A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n, \tag{35.14}$$

where $\{h_n\}$ and $\{k_n\}$ are orthonormal sets in H and K respectively and $\{\lambda_n\} \subset \mathbb{R}$ such that $\|A\|^2 = \sum |\lambda_n|^2 < \infty$.

Proof. Let $A \equiv \sum a_{ji} h_j \otimes k_i$, where $\{h_i\}$ and $\{k_j\}$ are orthonormal bases for H and K respectively and $\{a_{ji}\} \subset \mathbb{R}$ such that $\|A\|^2 = \sum |a_{ji}|^2 < \infty$. Then evidently, $T(A)k \equiv \sum a_{ji} h_j (k_i, k)$ and

$$\|T(A)k\|^2 = \sum_j |\sum_i a_{ji}(k_i, k)|^2 \leq \sum_j \sum_i |a_{ji}|^2 |(k_i, k)|^2 \leq \sum_j \sum_i |a_{ji}|^2 \|k\|^2.$$

Thus $T : H \otimes K \to B(K, H)$ is bounded. Moreover,

$$\|T(A)\|_{HS}^2 \equiv \sum \|T(A)k_i\|^2 = \sum_{ij} |a_{ji}|^2 = \|A\|^2,$$

which proves the T is an isometry.

We will now prove that T is surjective and at the same time prove Eq. (35.14). To motivate the construction, suppose that $Q = T(A)$ where A is given as in Eq. (35.14). Then

$$Q^*Q = T(\sum_{n=1}^{\infty} \lambda_n k_n \otimes h_n) T(\sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n) = T(\sum_{n=1}^{\infty} \lambda_n^2 k_n \otimes k_n).$$

That is $\{k_n\}$ is an orthonormal basis for $(\text{nul} Q^*Q)^\perp$ with $Q^*Qk_n = \lambda_n^2 k_n$. Also $Qk_n = \lambda_n h_n$, so that $h_n = \lambda_n^{-1} Qk_n$.

We will now reverse the above argument. Let $Q \in HS(K, H)$. Then Q^*Q is a self-adjoint compact operator on K. Therefore there is an orthonormal basis $\{k_n\}_{n=1}^{\infty}$

for the $(\text{nul}Q^*Q)^\perp$ which consists of eigenvectors of Q^*Q. Let $\lambda_n \in (0,\infty)$ such that $Q^*Qk_n = \lambda_n^2 k_n$ and set $h_n = \lambda_n^{-1}Qk_n$. Notice that

$$(h_n, h_m) = (\lambda_n^{-1}Qk_n, \lambda_m^{-1}Qk_m) = (\lambda_n^{-1}k_n, \lambda_m^{-1}Q^*Qk_m) = (\lambda_n^{-1}k_n, \lambda_m^{-1}\lambda_m^2 k_m) = \delta_{mn},$$

so that $\{h_n\}$ is an orthonormal set in H. Define

$$A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n$$

and notice that $T(A)k_n = \lambda_n h_n = Qk_n$ for all n and $T(A)k = 0$ for all $k \in \text{nul}Q = \text{nul}Q^*Q$. That is $T(A) = Q$. Therefore T is surjective and Eq. (35.14) holds. ■

Recall that $\sqrt{1-z} = 1 - \sum_{i=1}^{\infty} c_i z^i$ for $|z| < 1$, where $c^i \geq 0$ and $\sum_{i=1}^{\infty} c_i < \infty$. For an operator A on H such that $A \geq 0$ and $\|A\|_{B(H)} \leq 1$, the square root of A is given by

$$\sqrt{A} = I - \sum_{i=1}^{\infty} c_i (A-I)^i.$$

See Theorem VI.9 on p. 196 of Reed and Simon [**?**]. The next proposition is problem 14 and 15 on p. 217 of [**?**]. Let $|A| \equiv \sqrt{A^*A}$.

Proposition 35.30 (Square Root)**.** *Suppose that A_n and A are positive operators on H and $\|A - A_n\|_{B(H)} \to 0$ as $n \to \infty$, then $\sqrt{A_n} \to \sqrt{A}$ in $B(H)$ also. Moreover, A_n and A are general bounded operators on H and $A_n \to A$ in the operator norm then $|A_n| \to |A|$.*

Proof. With out loss of generality, assume that $\|A_n\| \leq 1$ for all n. This implies also that that $\|A\| \leq 1$. Then

$$\sqrt{A} - \sqrt{A_n} = \sum_{i=1}^{\infty} c_i \{(A_n - I)^i - (A-I)^i\}$$

and hence

$$\|\sqrt{A} - \sqrt{A_n}\| \leq \sum_{i=1}^{\infty} c_i \|(A_n - I)^i - (A-I)^i\|. \tag{35.15}$$

For the moment we will make the additional assumption that $A_n \geq \epsilon I$, where $\epsilon \in (0,1)$. Then $0 \leq I - A_n \leq (1-\epsilon)I$ and in particular $\|I - A_n\|_{B(H)} \leq (1-\epsilon)$.

Now suppose that Q, R, S, T are operators on H, then $QR - ST = (Q-S)R + S(R-T)$ and hence

$$\|QR - ST\| \leq \|Q - S\|\|R\| + \|S\|\|R - T\|.$$

Setting $Q = A_n - I$, $R \equiv (A_n - I)^{i-1}$, $S \equiv (A - I)$ and $T = (A-I)^{i-1}$ in this last inequality gives

$$\begin{aligned} \|(A_n - I)^i - (A-I)^i\| &\leq \|A_n - A\|\|(A_n - I)^{i-1}\| + \|(A-I)\|\|(A_n - I)^{i-1} - (A-I)^{i-1}\| \\ &\leq \|A_n - A\|(1-\epsilon)^{i-1} + (1-\epsilon)\|(A_n - I)^{i-1} - (A-I)^{i-1}\|. \end{aligned} \tag{35.16}$$

It now follows by induction that

$$\|(A_n - I)^i - (A-I)^i\| \leq i(1-\epsilon)^{i-1}\|A_n - A\|.$$

Inserting this estimate into (35.15) shows that

$$\|\sqrt{A} - \sqrt{A_n}\| \leq \sum_{i=1}^{\infty} c_i i (1-\epsilon)^{i-1}\|A_n - A\| = \frac{1}{2}\frac{1}{\sqrt{1-(1-\epsilon)}}\|A - A_n\| = \frac{1}{2}\frac{1}{\sqrt{\epsilon}}\|A - A_n\| \to 0.$$

Therefore we have shown if $A_n \geq \epsilon I$ for all n and $A_n \to A$ in norm then $\sqrt{A_n} \to \sqrt{A}$ in norm.

For the general case where $A_n \geq 0$, we find that for all $\epsilon > 0$

$$\lim_{n\to\infty} \sqrt{A_n + \epsilon} = \sqrt{A + \epsilon}. \tag{35.17}$$

By the spectral theorem[54]

$$\|\sqrt{A+\epsilon} - \sqrt{A}\| \leq \max_{x\in\sigma(A)} |\sqrt{x+\epsilon} - \sqrt{x}| \leq \max_{0\leq x\leq \|A\|} |\sqrt{x+\epsilon} - \sqrt{x}| \to 0 \text{ as } \epsilon \to 0.$$

Since the above estimates are uniform in $A \geq 0$ such that $\|A\|$ is bounded, it is now an easy matter to conclude that Eq. (35.17) holds even when $\epsilon = 0$. ■

Now suppose that $A_n \to A$ in $B(H)$ and A_n and A are general operators. Then $A_n^* A_n \to A^* A$ in $B(H)$. So by what we have already proved,

$$|A_n| \equiv \sqrt{A_n^* A_n} \to |A| \equiv \sqrt{A^* A} \text{ in } B(H) \text{ as } n \to \infty.$$

Notation 35.31. In the future we will identify $A \in H \otimes K$ with $T(A) \in HS(K, H)$ and drop T from the notation. So that with this notation we have $(h \otimes k)k' = (k, k')h$.

Let $A \in H \otimes H$, we set $\|A\|_1 \equiv \text{tr}\sqrt{A^*A} \equiv \text{tr}\sqrt{T(A)^*T(A)}$ and we let

$$H \otimes_1 H \equiv \{A \in H \otimes H : \|A\|_1 < \infty\}.$$

We will now compute $\|A\|_1$ for $A \in H \otimes H$ described as in Eq. (35.14). First notice that $A^* = \sum_{n=1}^\infty \lambda_n k_n \otimes h_n$ and

$$A^*A = \sum_{n=1}^\infty \lambda_n^2 k_n \otimes k_n.$$

Hence $\sqrt{A^*A} = \sum_{n=1}^\infty |\lambda_n| k_n \otimes k_n$ and hence $\|A\|_1 = \sum_{n=1}^\infty |\lambda_n|$. Also notice that $\|A\|^2 = \sum_{n=1}^\infty |\lambda_n|^2$ and $\|A\|_{op} = \max_n |\lambda_n|$. Since

$$\|A\|_1^2 = \{\sum_{n=1}^\infty |\lambda_n|\}^2 \geq \sum_{n=1}^\infty |\lambda_n|^2 = \|A\|^2,$$

we have the following relations among the various norms,

$$\|A\|_{op} \leq \|A\| \leq \|A\|_1. \tag{35.18}$$

Proposition 35.32. *There is a continuous linear map $C : H \otimes_1 H \to \mathbb{R}$ such that $C(h \otimes k) = (h, k)$ for all $h, k \in H$. If $A \in H \otimes_1 H$, then*

$$CA = \sum (e_m \otimes e_m, A), \tag{35.19}$$

where $\{e_m\}$ is any orthonormal basis for H. Moreover, if $A \in H \otimes_1 H$ is positive, i.e. $T(A)$ is a non-negative operator, then $\|A\|_1 = CA$.

---

[54] It is possible to give a more elementary proof here. Indeed, assume further that $\|A\| \leq \alpha < 1$, then for $\epsilon \in (0, 1-\alpha)$, $\|\sqrt{A+\epsilon} - \sqrt{A}\| \leq \sum_{i=1}^\infty c_i \|(A+\epsilon)^i - A^i\|$. But

$$\|(A+\epsilon)^i - A^i\| \leq \sum_{k=1}^i \binom{i}{k} \epsilon^k \|A^{i-k}\| \leq \sum_{k=1}^i \binom{i}{k} \epsilon^k \|A\|^{i-k} = (\|A\| + \epsilon)^i - \|A\|^i,$$

so that $\|\sqrt{A+\epsilon} - \sqrt{A}\| \leq \sqrt{\|A\| + \epsilon} - \sqrt{\|A\|} \to 0$ as $\epsilon \to 0$ uniformly in $A \geq 0$ such that $\|A\| \leq \alpha < 1$.

Proof. Let $A \in H \otimes_1 H$ be given as in Eq. (35.14) with $\sum_{n=1}^{\infty} |\lambda_n| = \|A\|_1 < \infty$. Then define $CA \equiv \sum_{n=1}^{\infty} \lambda_n (h_n, k_n)$ and notice that $|CA| \le \sum |\lambda_n| = \|A\|_1$, which shows that C is a contraction on $H \otimes_1 H$. (Using the universal property of $H \otimes_f H$ it is easily seen that C is well defined.) Also notice that for $M \in \mathbb{Z}_+$ that

$$\sum_{m=1}^{M} (e_m \otimes e_m, A) = \sum_{n=1}^{\infty} \sum_{m=1}^{M} (e_m \otimes e_m, \lambda_n h_n \otimes k_n,), \tag{35.20}$$

$$= \sum_{n=1}^{\infty} \lambda_n (P_M h_n, k_n), \tag{35.21}$$

where P_M denotes orthogonal projection onto $\operatorname{span}\{e_m\}_{m=1}^{M}$. Since $|\lambda_n (P_M h_n, k_n)| \le |\lambda_n|$ and $\sum_{n=1}^{\infty} |\lambda_n| = \|A\|_1 < \infty$, we may let $M \to \infty$ in Eq. (35.21) to find that

$$\sum_{m=1}^{\infty} (e_m \otimes e_m, A) = \sum_{n=1}^{\infty} \lambda_n (h_n, k_n) = CA.$$

This proves Eq. (35.19).

For the final assertion, suppose that $A \ge 0$. Then there is an orthonormal basis $\{k_n\}_{n=1}^{\infty}$ for the $(\mathrm{nul} A)^{\perp}$ which consists of eigenvectors of A. That is $A = \sum \lambda_n k_n \otimes k_n$ and $\lambda_n \ge 0$ for all n. Thus $CA = \sum \lambda_n$ and $\|A\|_1 = \sum \lambda_n$.

Proposition 35.33 (Noncommutative Fatou' s Lemma)**.** *Let A_n be a sequence of positive operators on a Hilbert space H and $A_n \to A$ weakly as $n \to \infty$, then*

$$trA \le \liminf_{n\to\infty} trA_n. \tag{35.22}$$

Also if $A_n \in H \otimes_1 H$ and $A_n \to A$ in $B(H)$, then

$$\|A\|_1 \le \liminf_{n\to\infty} \|A_n\|_1. \tag{35.23}$$

■

Proof. Let A_n be a sequence of positive operators on a Hilbert space H and $A_n \to A$ weakly as $n \to \infty$ and $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Then by Fatou's lemma for sums,

$$\begin{aligned} \mathrm{tr}A = \sum_{k=1}^{\infty} (Ae_k, e_k) &= \sum_{k=1}^{\infty} \lim_{n\to\infty} (A_n e_k, e_k) \\ &\le \liminf_{n\to\infty} \sum_{k=1}^{\infty} (A_n e_k, e_k) = \liminf_{n\to\infty} \mathrm{tr}A_n. \end{aligned}$$

Now suppose that $A_n \in H \otimes_1 H$ and $A_n \to A$ in $B(H)$. Then by Proposition 35.30, $|A_n| \to |A|$ in $B(H)$ as well. Hence by Eq. (35.22), $\|A\|_1 \equiv \mathrm{tr}|A| \le \liminf_{n\to\infty} \mathrm{tr}|A_n| \le \liminf_{n\to\infty} \|A_n\|_1$. ■

Proposition 35.34. *Let X be a Banach space, $B : H \times K \to X$ be a bounded bi-linear form, and $\|B\| \equiv \sup\{|B(h,k)| : \|h\|\|k\| \le 1\}$. Then there is a unique bounded linear map $\tilde{B} : H \otimes_1 K \to X$ such that $\tilde{B}(h \otimes k) = B(h,k)$. Moreover $\|\tilde{B}\|_{op} = \|\tilde{B}\|$.*

Proof. Let $A=\sum_{n=1}^\infty \lambda_n h_n\otimes k_n\in H\otimes_1 K$ as in Eq. (35.14). Clearly, if $\tilde{B}$ is to exist we must have $\tilde{B}(A)\equiv\sum_{n=1}^\infty \lambda_n B(h_n,k_n)$. Notice that

$$\sum_{n=1}^\infty |\lambda_n||B(h_n,k_n)|\le\sum_{n=1}^\infty |\lambda_n|\|B\|=\|A\|_1\cdot\|B\|.$$

This shows that $\tilde{B}(A)$ is well defined and that $\|\tilde{B}\|_{op}\le\|\tilde{B}\|$. The opposite inequality follows from the trivial computation:

$$\|B\|=\sup\{|B(h,k)|:\|h\|\|k\|=1\}=\sup\{|\tilde{B}(h\otimes k)|:\|h\otimes_1 k\|_1=1\}\le\|\tilde{B}\|_{op}.$$

■

Lemma 35.35. *Suppose that $P\in B(H)$ and $Q\in B(K)$, then $P\otimes Q:H\otimes K\to H\otimes K$ is a bounded operator. Moreover, $P\otimes Q(H\otimes_1 K)\subset H\otimes_1 K$ and we have the norm equalities*

$$\|P\otimes Q\|_{B(H\otimes K)}=\|P\|_{B(H)}\|Q\|_{B(K)}$$

and

$$\|P\otimes Q\|_{B(H\otimes_1 K)}=\|P\|_{B(H)}\|Q\|_{B(K)}.$$

We will give essentially the same proof of $\|P\otimes Q\|_{B(H\otimes K)}=\|P\|_{B(H)}\|Q\|_{B(K)}$ as the proof on p. 299 of Reed and Simon [**?**]. Let $A\in H\otimes K$ as in Eq. (35.14). Then

$$(P\otimes I)A=\sum_{n=1}^\infty \lambda_n Ph_n\otimes k_n$$

and hence

$$(P\otimes I)A\{(P\otimes I)A\}^*=\sum_{n=1}^\infty \lambda_n^2 Ph_n\otimes Ph_n.$$

Therefore,

$$\begin{aligned}\|(P\otimes I)A\|^2&=\operatorname{tr}(P\otimes I)A\{(P\otimes I)A\}^*\\&=\sum_{n=1}^\infty\lambda_n^2(Ph_n,Ph_n)\le\|P\|^2\sum_{n=1}^\infty\lambda_n^2\\&=\|P\|^2\|A\|_1^2,\end{aligned}$$

which shows that Thus $\|P\otimes I\|_{B(H\otimes K)}\le\|P\|$. By symmetry, $\|I\otimes Q\|_{B(H\otimes K)}\le\|Q\|$. Since $P\otimes Q=(P\otimes I)(I\otimes Q)$, we have

$$\|P\otimes Q\|_{B(H\otimes K)}\le\|P\|_{B(H)}\|Q\|_{B(K)}.$$

The reverse inequality is easily proved by considering $P\otimes Q$ on elements of the form $h\otimes k\in H\otimes K$.

Proof. Now suppose that $A\in H\otimes_1 K$ as in Eq. (35.14). Then

$$\|(P\otimes Q)A\|_1\le\sum_{n=1}^\infty|\lambda_n|\|Ph_n\otimes Qk_n\|_1\le\|P\|\|Q\|\sum_{n=1}^\infty|\lambda_n|=\|P\|\|Q\|\|A\|,$$

which shows that

$$\|P\otimes Q\|_{B(H\otimes_1 K)}\le\|P\|_{B(H)}\|Q\|_{B(K)}.$$

Again the reverse inequality is easily proved by considering $P\otimes Q$ on elements of the form $h\otimes k\in H\otimes_1 K$. ■

Lemma 35.36. *Suppose that P_m and Q_m are orthogonal projections on H and K respectively which are strongly convergent to the identity on H and K respectively. Then $P_m \otimes Q_m : H \otimes_1 K \longrightarrow H \otimes_1 K$ also converges strongly to the identity in $H \otimes_1 K$.*

Proof. Let $A = \sum_{n=1}^{\infty} \lambda_n h_n \otimes k_n \in H \otimes_1 K$ as in Eq. (35.14). Then

$$\begin{aligned}
\|P_m \otimes Q_m A - A\|_1 &\leq \sum_{n=1}^{\infty} |\lambda_n| \|P_m h_n \otimes Q_m k_n - h_n \otimes k_n\|_1 \\
&= \sum_{n=1}^{\infty} |\lambda_n| \|(P_m h_n - h_n) \otimes Q_m k_n + h_n \otimes (Q_m k_n - k_n)\|_1 \\
&\leq \sum_{n=1}^{\infty} |\lambda_n| \{\|P_m h_n - h_n\| \|Q_m k_n\| + \|h_n\| \|Q_m k_n - k_n\|\} \\
&\leq \sum_{n=1}^{\infty} |\lambda_n| \{\|P_m h_n - h_n\| + \|Q_m k_n - k_n\|\} \to 0 \text{ as } m \to \infty
\end{aligned}$$

by the dominated convergence theorem. ■