2. Math 240A (Driver) Homework #2: Due October 6, 2000

Hand in all problems in this problem set.
For these problems, let $X$ be a set and $\mathcal{M} = \mathcal{P}(X)$ be the “$\sigma$-algebra” consisting of all subsets of $X$.

2.1. Measure Problems.

(1) (** Suppose that $\mu_n : \mathcal{M} \to [0, \infty]$ are measures on $\mathcal{M}$ for $n \in \mathbb{N}$. Also suppose that $\mu_n(A)$ is increasing in $n$ for all $A \in \mathcal{M}$. Prove that $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(A) := \lim_{n \to \infty} \mu_n(A)$ is also a measure.

(2) (** Now suppose that $\Lambda$ is some index set and for each $\lambda \in \Lambda$, $\mu_\lambda : \mathcal{M} \to [0, \infty]$ is a measure on $\mathcal{M}$. Define $\mu : \mathcal{M} \to [0, \infty]$ by $\mu(A) = \sum_{\lambda \in \Lambda} \mu_\lambda(A)$ for each $A \in \mathcal{M}$. Show that $\mu$ is also a measure.

**Corollary 2.1.** Suppose that $\lambda : X \to [0, \infty]$ is a function. Define $\mu : \mathcal{M} \to [0, \infty]$ by

$$\mu(A) = \sum_{x \in A} \lambda(x).$$

Then $\mu$ is a measure on $\mathcal{M}$.

**Proof.** For $x \in X$, let $\delta_x : \mathcal{M} \to [0, \infty)$ be defined by

$$\delta_x(A) = 1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$ 

It is easy to check that $\delta_x$ is a measure as well as $\lambda(x)\delta_x$. Hence by Problem 2.1.2 above we learn that

$$\mu(A) = \sum_{x \in X} \lambda(x) \delta_x(A) = \sum_{x \in A} \lambda(x)$$

is also a measure. □

2.2. Dominated Convergence Theorem Problems.

(1) (** Suppose $V \subset \mathbb{R}^n$ is an open set, $t_0 \in V$, and $G : V \setminus \{t_0\} \to \mathbb{C}$ is a function on $V \setminus \{t_0\}$. Show that $\lim_{t \to t_0} G(t)$ exists and is equal to $A \in \mathbb{C}$, iff for all sequences $\{t_n\}_{n=1}^{\infty} \subset V \setminus \{t_0\}$ which converge to $t_0$ (i.e. $\lim_{n \to \infty} t_n = t_0$) we have

$$\lim_{n \to \infty} G(t_n) = A.$$

(2) (** Suppose that $X$ is a set, $V \subset \mathbb{R}^n$ is an open set, and $f : V \times X \to \mathbb{C}$ is a function satisfying:

(a) For each $x \in X$, the function $t \to f(t, x)$ is continuous on $V$.

(b) There is a summable function $g : X \to [0, \infty)$ such that

$$|f(t, x)| \leq g(x)$$

for all $x \in X$ and $t \in V$. Show that

$$F(t) := \sum_{x \in X} f(t, x)$$

is a continuous function for $t \in V$.

(3) (** Suppose that $X$ is a set, $J = (a, b) \subset \mathbb{R}$ is an interval, and $f : J \times X \to \mathbb{C}$ is a function satisfying:

(a) For each $x \in X$, the function $t \to f(t, x)$ is differentiable on $J$,
(b) There is a summable function $g : X \to [0, \infty)$ such that
\[
\left| f(t, x) \right| := \left| \frac{d}{dt} f(t, x) \right| \leq g(x) \text{ for all } x \in X.
\]

(c) There is a $t_0 \in J$ such that $\sum_{x \in X} |f(t_0, x)| < \infty$.

Show:

A: for all $t \in J$ that
\[
\sum_{x \in X} |f(t, x)| < \infty.
\]

B: Let $F(t) := \sum_{x \in X} f(t, x)$, show $F$ is differentiable on $J$ and that
\[
\dot{F}(t) = \sum_{x \in X} f(t, x).
\]
(Hint: Use the mean value theorem.)

(4) \text{(**)} Let $\{a_n\}_{n=-\infty}^{\infty}$ be a summable sequence of complex numbers, i.e.
\[
\sum_{n=-\infty}^{\infty} |a_n| < \infty.
\]
For $t \geq 0$ and $x \in \mathbb{R}$, define
\[
f(t, x) = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2} e^{inx},
\]
where as usual $e^{ix} = \cos(x) + i\sin(x)$. Prove the following facts about $f$:
(a) $f(t, x)$ is continuous for $(t, x) \in [0, \infty) \times \mathbb{R}$.
(b) $\partial f(t, x)/\partial t$, $\partial f(t, x)/\partial x$ and $\partial^2 f(t, x)/\partial x^2$ exist for $t > 0$ and $x \in \mathbb{R}$.
(c) $f$ satisfies the heat equation, namely
\[
\partial f(t, x)/\partial t = \partial^2 f(t, x)/\partial x^2 \text{ for } t > 0 \text{ and } x \in \mathbb{R}.
\]

2.3. Metric Spaces.

(1) \text{(**)} Let $X = C([0, 1], \mathbb{R})$ and for $f, g \in X$ define
\[
\rho(f, g) := \int_0^1 |f(t) - g(t)| \, dt,
\]
where the integral is the Riemann integral. Show that $(X, \rho)$ is a metric space and show by example that this metric space is \textbf{not} complete.