2. Math 240B (Driver) Homework #2: Due Friday January 26, 2001

Only hand in the problems with an asterisk after them. However, you should make sure that you are able to do all of the problems.

Folland, Chapter 4: 2, 3*, 5, 6*, 7, 11*, 15*, 16*, 17*, 18, 23*, 26*

Hint for #23: The complement of $A$ is a countable union of disjoint open intervals. Extend $f$ on each of these intervals to be a linear function.

2.1. Metric Space Problems. Let $(X, \rho)$ be a metric space.

(1) Suppose that $\{x_n\}$ is Cauchy, show there is a subsequence $y_j \equiv x_{n_j}$ such that $\sum_{j=1}^{\infty} \rho(y_{j+1}, y_j) < \infty$.

(2) Suppose that $\{x_n\}_{n=1}^{\infty} \subset X$ is a sequence such that $\sum_{n=1}^{\infty} \rho(x_{n+1}, x_n) < \infty$, show that $\{x_n\}$ is Cauchy.

(3) Show that if $\{x_n\}_{n=1}^{\infty}$ is Cauchy and there is exists a subsequence $y_j \equiv x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j \to \infty} y_j$ exists, then $\lim_{n \to \infty} x_n$ also exists and is equal to $x$.

(4) (*) Show that $(X, \rho)$ is complete iff for ever sequence $\{x_n\}$ satisfying $\sum_{n=1}^{\infty} \rho(x_{n+1}, x_n) < \infty$ is necessarily convergent.

(5) (*) Contraction Mapping Principle Suppose now that $(X, \rho)$ is complete, $T : X \to X$ is a map and there exists $\alpha \in (0, 1)$ such that $\rho(T(x), T(y)) \leq \alpha \rho(x, y)$ for all $x, y \in X$. Prove that $T$ has a fixed point, i.e. there is a unique element $x \in X$ such that $T(x) = x$.

(Notice that this fixed point is unique since if $x = T(x)$ and $y = T(y)$, then $\rho(x, y) = \rho(T(x), T(y)) \leq \alpha \rho(x, y)$ and therefore $\rho(x, y)(1 - \alpha) \leq 0$. This shows that $\rho(x, y) = 0$, i.e. that $x = y$.)

Hint: Let $x_0 \in X$ be arbitrary and define $x_n$ inductively by $x_{n+1} = T(x_n)$. Then show that $\rho(x_{n+1}, x_n) \leq C \alpha^n$ where $C$ is a finite constant. Use the above problems to conclude that $x \equiv \lim_{n \to \infty} x_n$ exists. Then notice that

$$\rho(x, x_n) = \lim_{m \to \infty} \rho(x_m, x_n) \leq \lim_{m \to \infty} \sum_{k=n}^{m-1} \rho(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} \alpha^k = \frac{\alpha^n}{1 - \alpha}.$$