8.4. **Exercises.**

Exercise 8.8. Let (X,τ) be a topological space, $A\subset X$, $i_A : A\to X$ be the inclusion map and $\tau_A := i_A^{-1}(\tau)$ be the relative topology on A. Verify $\tau_A=\{A\cap V : V\in\tau\}$ and show $C\subset A$ is closed in (A,τ_A) iff there exists a closed set $F\subset X$ such that $C=A\cap F$. (If you get stuck, see the remarks after Definition 3.17 where this has already been proved.)

Exercise 8.9. Let (X,τ) and (Y,τ') be a topological spaces, $f : X\to Y$ be a function, $\mathcal{U}$ be an open cover of X and $\{F_j\}_{j=1}^n$ be a finite cover of X by closed sets.

1. If $A\subset X$ is any set and $f : X\to Y$ is (τ,τ') – continuous then $f|_A : A\to Y$ is (τ_A,τ') – continuous.
2. Show $f : X\to Y$ is (τ,τ') – continuous iff $f|_U : U\to Y$ is (τ_U,τ') – continuous for all $U\in\mathcal{U}$.
3. Show $f : X\to Y$ is (τ,τ') – continuous iff $f|_{F_j} : F_j\to Y$ is (τ_{F_j},τ') – continuous for all $j=1,2,\dots,n$.
4. (A baby form of the Tietze extension Theorem.) Suppose $V\in\tau$ and $f : V\to\mathbb{C}$ is a continuous function such supp$(f)\subset V$, then $F : X\to\mathbb{C}$ defined by
$$F(x)=\begin{cases} f(x) & \text{if} \quad x\in V\\ 0 & \text{otherwise}\end{cases}$$
is continuous.

Exercise 8.10. Prove Theorem 8.16. **Hints:**

1. By Proposition 8.13, there exists a precompact open set V such that $K\subset V\subset\bar V\subset U$. Now suppose that $f : K\to[0,\alpha]$ is continuous with $\alpha\in(0,1]$ and let $A := f^{-1}([0,\frac{1}{3}\alpha])$ and $B := f^{-1}([\frac{2}{3}\alpha,1])$. Appeal to Lemma 8.15 to find a function $g\in C(X,[0,\alpha/3])$ such that $g=\alpha/3$ on B and supp$(g)\subset V\setminus A$.
2. Now follow the argument in the proof of Theorem 8.2 to construct $F\in C(X,[a,b])$ such that $F|_K=f$.
3. For $c\in[a,b]$, choose $\phi\prec U$ such that $\phi=1$ on K and replace F by $F_c := \phi F+(1-\phi)c$.

Exercise 8.11 (Sterographic Projection)**.** Let $X=\mathbb{R}^n$, $X^* := X\cup\{\infty\}$ be the one point compactification of X, $S^n := \{y\in\mathbb{R}^{n+1} : |y|=1\}$ be the unit sphere in $\mathbb{R}^{n+1}$ and $N=(0,\dots,0,1)\in\mathbb{R}^{n+1}$. Define $f : S^n\to X^*$ by $f(N)=\infty$, and for $y\in S^n\setminus\{N\}$ let $f(y)=b\in\mathbb{R}^n$ be the unique point such that $(b,0)$ is on the line containing N and y, see Figure 22 below. Find a formula for f and show $f : S^n\to X^*$ is a homeomorphism. (So the one point compactification of $\mathbb{R}^n$ is homeomorphic to the n sphere.)

Exercise 8.12. Let (X,τ) be a locally compact Hausdorff space. Show (X,τ) is separable iff (X^*,τ^*) is separable.

Exercise 8.13. Show by example that there exists a locally compact metric space (X,d) such that the one point compactification, $(X^* := X\cup\{\infty\},\tau^*)$, is **not** metrizable. **Hint**: use exercise 8.12.

Exercise 8.14. Suppose (X,d) is a locally compact and σ – compact metric space. Show the one point compactification, $(X^* := X\cup\{\infty\},\tau^*)$, is metrizable.

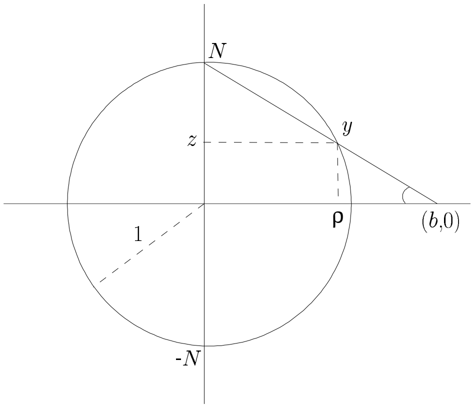

FIGURE 22. Sterographic projection and the one point compactification of $\mathbb{R}^n$.

9. APPROXIMATION THEOREMS AND CONVOLUTIONS

Let $(X, \mathcal{M}, \mu)$ be a measure space, $\mathcal{A} \subset \mathcal{M}$ an algebra.

Notation 9.1. Let $\mathbb{S}_f(\mathcal{A}, \mu)$ denote those simple functions $\phi : X \to \mathbb{C}$ such that $\phi^{-1}(\{\lambda\}) \in \mathcal{A}$ for all $\lambda \in \mathbb{C}$ and $\mu(\phi \neq 0) < \infty$.

For $\phi \in \mathbb{S}_f(\mathcal{A}, \mu)$ and $p \in [1, \infty)$, $|\phi|^p = \sum_{z \neq 0} |z|^p 1_{\{\phi = z\}}$ and hence

$$\int |\phi|^p \, d\mu = \sum_{z \neq 0} |z|^p \mu(\phi = z) < \infty$$

so that $\mathbb{S}_f(\mathcal{A}, \mu) \subset L^p(\mu)$.

Lemma 9.2 (Simple Functions are Dense)**.** *The simple functions, $\mathbb{S}_f(\mathcal{M}, \mu)$, form a dense subspace of $L^p(\mu)$ for all $1 \leq p < \infty$.*

Proof. Let $\{\phi_n\}_{n=1}^\infty$ be the simple functions in the approximation Theorem 5.12. Since $|\phi_n| \leq |f|$ for all n, $\phi_n \in \mathbb{S}_f(\mathcal{M}, \mu)$ (verify!) and

$$|f - \phi_n|^p \leq (|f| + |\phi_n|)^p \leq 2^p |f|^p \in L^1.$$

Therefore, by the dominated convergence theorem,

$$\lim_{n \to \infty} \int |f - \phi_n|^p d\mu = \int \lim_{n \to \infty} |f - \phi_n|^p d\mu = 0.$$

■

Theorem 9.3 (Separable Algebras implies Separability of L^p – Spaces)**.** *Suppose $1 \leq p < \infty$ and $\mathcal{A} \subset \mathcal{M}$ is an algebra such that $\sigma(\mathcal{A}) = \mathcal{M}$ and μ is σ-finite on $\mathcal{A}$. Then $\mathbb{S}_f(\mathcal{A}, \mu)$ is dense in $L^p(\mu)$. Moreover, if $\mathcal{A}$ is countable, then $L^p(\mu)$ is separable and*

$$\mathbb{D} = \{\sum a_j 1_{A_j} : a_j \in \mathbb{Q} + i\mathbb{Q},\ A_j \in \mathcal{A} \text{ with } \mu(A_j) < \infty\}$$

is a countable dense subset.

Proof. First Proof. Let $X_k \in \mathcal{A}$ be sets such that $\mu(X_k) < \infty$ and $X_k \uparrow X$ as $k \to \infty$. For $k \in \mathbb{N}$ let $\mathcal{H}_k$ denote those bounded $\mathcal{M}$ – measurable functions, f, on X such that $1_{X_k} f \in \overline{\mathbb{S}_f(\mathcal{A}, \mu)}^{L^p(\mu)}$. It is easily seen that $\mathcal{H}_k$ is a vector space closed