Conclude from this that
\[ I(f) = \lim_{|\pi| \to 0} \sum_{i=0}^{n-1} f(c_i)(F(t_{i+1}) - F(t_i)). \]

As usual we will write this integral as \( \int_{-M}^M f dF \) and as \( \int_{-M}^M f(t) dt \) if \( F(t) = t \).

**Exercise 11.5.** Folland problem 1.28.

**Exercise 11.6.** Suppose that \( F \in C^1(\mathbb{R}) \) is an increasing function and \( \mu_F \) is the unique Borel measure on \( \mathbb{R} \) such that \( \mu_F((a,b]) = F(b) - F(a) \) for all \( a \leq b \). Show that \( d\mu_F = \rho dm \) for some function \( \rho \geq 0 \). Find \( \rho \) explicitly in terms of \( F \).

**Exercise 11.7.** Suppose that \( F(x) = e1_{x \geq 3} + \pi 1_{x \geq 7} \) and \( \mu_F \) is the is the unique Borel measure on \( \mathbb{R} \) such that \( \mu_F((a,b]) = F(b) - F(a) \) for all \( a \leq b \). Give an explicit description of the measure \( \mu_F \).

**Exercise 11.8.** Let \( E \in \mathcal{B}_\mathbb{R} \) with \( m(E) > 0 \). Then for any \( \alpha \in (0, 1) \) there exists an open interval \( J \subset \mathbb{R} \) such that \( m(E \cap J) \geq \alpha m(J) \). **Hints:** 1. Reduce to the case where \( m(E) \in (0, \infty) \). 2) Approximate \( E \) from the outside by an open set \( V \subset \mathbb{R} \). 3. Make use of Exercise 3.43, which states that \( V \) may be written as a disjoint union of open intervals.

11.10.1. **The Laws of Large Number Exercises.** For the rest of the problems of this section, let \( \nu \) be a probability measure on \( \mathcal{B}_\mathbb{R} \) such that \( \int_{\mathbb{R}} |x| d\nu(x) < \infty \), \( \mu_\nu := \nu \) for \( n \in \mathbb{N} \) and \( \mu \) denote the infinite product measure as constructed in Corollary 11.40. So \( \mu \) is the unique measure on \( (X := \mathbb{R}^N, \mathcal{B} := \mathcal{B}_{\mathbb{R}^N}) \) such that
\[
(11.43) \quad \int_X f(x_1, x_2, \ldots, x_N) d\mu(x) = \int_{\mathbb{R}^N} f(x_1, x_2, \ldots, x_N) d\nu(x_1) \ldots d\nu(x_N)
\]
for all \( N \in \mathbb{N} \) and bounded measurable functions \( f : \mathbb{R}^N \to \mathbb{R} \). We will also use the following notation:
\[
S_n(x) := \frac{1}{n} \sum_{k=1}^{n} x_k \text{ for } x \in X,
\]
\[
m := \int_{\mathbb{R}} x d\nu(x) \text{ the average of } \nu,
\]
\[
\sigma^2 := \int_{\mathbb{R}} (x - m)^2 d\nu(x) \text{ the variance of } \nu \text{ and}
\]
\[
\gamma := \int_{\mathbb{R}} (x - m)^4 d\nu(x).
\]
The variance may also be written as \( \sigma^2 = \int_{\mathbb{R}} x^2 d\nu(x) - m^2 \).

**Exercise 11.9** (Weak Law of Large Numbers). Suppose further that \( \sigma^2 < \infty \), show \( \int_X S_n d\mu = m \),
\[
||S_n - m||_2^2 = \int_X (S_n - m)^2 d\mu = \frac{\sigma^2}{n}
\]
and \( \mu(|S_n - m| > \epsilon) \leq \frac{\sigma^2}{\epsilon^2 m^2} \) for all \( \epsilon > 0 \) and \( n \in \mathbb{N} \).
Exercise 11.10 (A simple form of the Strong Law of Large Numbers). Suppose now that $\gamma := \int_{\mathbb{R}} (x - m)^4 \, d\nu(x) < \infty$. Show for all $\epsilon > 0$ and $n \in \mathbb{N}$ that
\[
\|S_n - m\|_4^4 = \int_X (S_n - m)^4 \, d\mu = \frac{1}{n^4} \left(n \gamma + 3(n - 1)\sigma^4\right)
\]
and
\[
\mu(|S_n - m| > \epsilon) \leq \frac{n^{-1} \gamma + 3(1 - n^{-1})\sigma^4}{\epsilon^4 n^2}.
\]
Conclude from the last estimate and the first Borel Cantelli Lemma 5.22 that $\lim_{n \to \infty} S_n(x) = m$ for $\mu$-a.e. $x \in X$.

Exercise 11.11. Suppose $\gamma := \int_{\mathbb{R}} (x - m)^4 \, d\nu(x) < \infty$ and $m = \int_{\mathbb{R}} (x - m) \, d\nu(x) \neq 0$. For $\lambda > 0$ let $T_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T_\lambda(x) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots)$, $\mu_\lambda = \mu \circ T_\lambda^{-1}$ and
\[
X_\lambda := \left\{ x \in \mathbb{R}^n : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n x_j = \lambda \right\}.
\]
Show
\[
\mu_\lambda(X_{\lambda'}) = \delta_{\lambda, \lambda'} = \begin{cases} 
1 & \text{if } \lambda = \lambda' \\
0 & \text{if } \lambda \neq \lambda'
\end{cases}
\]
and use this to show if $\lambda \neq 1$, then $d\mu_\lambda \neq \rho \, d\mu$ for any measurable function $\rho : \mathbb{R}^n \to [0, \infty]$. 