Remark 9.54. Given any collection of bounded real valued functions $\mathcal{F}$ on X, let $\mathcal{H}(\mathcal{F})$ be the subspace of $B(X,\mathbb{R})$ generated by $\mathcal{F}$, i.e. $\mathcal{H}(\mathcal{F})$ is the smallest subspace of $B(X,\mathbb{R})$ which is closed under bounded convergence and contains $\mathcal{F}$. With this notation, Theorem 9.52 may be stated as follows. If $\mathcal{F}$ is a multiplicative system then $\mathcal{H}(\mathcal{F}) = B_{\sigma(\mathcal{F})}(X,\mathbb{R})$ – the space of bounded $\sigma(\mathcal{F})$ – measurable real valued functions on X.

9.6. **Exercises.**

Exercise 9.4. Let (X,τ) be a topological space, μ a measure on $\mathcal{B}_X = \sigma(\tau)$ and $f : X \to \mathbb{C}$ be a measurable function. Letting ν be the measure, $d\nu = |f|\, d\mu$, show $\mathrm{supp}(\nu) = \mathrm{supp}_\mu(f)$, where $\mathrm{supp}(\nu)$ is defined in Definition 7.40).

Exercise 9.5. Let (X,τ) be a topological space, μ a measure on $\mathcal{B}_X = \sigma(\tau)$ such that $\mathrm{supp}(\mu) = X$ (see Definition 7.40). Show $\mathrm{supp}_\mu(f) = \mathrm{supp}(f) = \overline{\{f \neq 0\}}$ for all $f \in C(X)$.

Exercise 9.6. Prove Proposition 9.23 by appealing to Corollary 5.43.

Exercise 9.7 (Integration by Parts)**.** Suppose that $(x,y) \in \mathbb{R}\times\mathbb{R}^{n-1} \to f(x,y) \in \mathbb{C}$ and $(x,y) \in \mathbb{R}\times\mathbb{R}^{n-1} \to g(x,y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^{n-1}$, $x \to f(x,y)$ and $x \to g(x,y)$ are continuously differentiable. Also assume $f\cdot g$, $\partial_x f \cdot g$ and $f\cdot \partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R}\times\mathbb{R}^{n-1}$, where $\partial_x f(x,y) := \frac{d}{dt} f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R}\times\mathbb{R}^{n-1}} \partial_x f(x,y)\cdot g(x,y)dxdy = -\int_{\mathbb{R}\times\mathbb{R}^{n-1}} f(x,y)\cdot \partial_x g(x,y)dxdy. \tag{9.27}$$

(Note: this result and Fubini's theorem proves Lemma 9.25.)

Hints: Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0 \in \mathbb{R}$ and set $\psi_\epsilon(x) = \psi(\epsilon x)$. First verify Eq. (9.27) with $f(x,y)$ replaced by $\psi_\epsilon(x) f(x,y)$ by doing the x – integral first. Then use the dominated convergence theorem to prove Eq. (9.27) by passing to the limit, $\epsilon \downarrow 0$.

Exercise 9.8. Let $M < \infty$, show there are polynomials $p_n(t)$ such that

$$\lim_{n\to\infty} \sup_{|t|\le M} \left||t| - p_n(t)\right| = 0$$

as follows. Let $f(t) = \sqrt{1-t}$ for $|t| \le 1$. By Taylor's theorem with integral remainder (see Eq. A.15 of Appendix A) or by analytic function theory, there are constants[23] $\alpha_n > 0$ for $n \in \mathbb{N}$ such that

$$\sqrt{1-x} = 1 - \sum_{n=1}^\infty \alpha_n x^n \text{ for all } |x| < 1.$$

Use this to prove $\sum_{n=1}^\infty \alpha_n = 1$ and therefore $q_m(x) := 1 - \sum_{n=1}^m \alpha_n x^n$

$$\lim_{m\to\infty} \sup_{|x|\le 1} \left|\sqrt{1-x} - q_m(x)\right| = 0.$$

Let $1 - x = t^2/M^2$, i.e. $x = 1 - t^2/M^2$, then

$$\lim_{m\to\infty} \sup_{|t|\le M} \left|\frac{|t|}{M} - q_m(1 - t^2/M^2)\right| = 0$$

so that $p_m(t) := M q_m(1 - t^2/M^2)$ are the desired polynomials.

[23] In fact $\alpha_n := \frac{(2n-3)!!}{2^n n!}$, but this is not needed.

Exercise 9.9. Given a continuous function $f : \mathbb{R} \to \mathbb{C}$ which is 2π -periodic and $\epsilon > 0$. Show there exists a trigonometric polynomial, $p(\theta) = \sum_{n=-N}^{n} \alpha_n e^{in\theta}$, such that $|f(\theta) - P(\theta)| < \epsilon$ for all $\theta \in \mathbb{R}$. **Hint:** show that there exists a unique function $F \in C(S^1)$ such that $f(\theta) = F(e^{i\theta})$ for all $\theta \in \mathbb{R}$.

Remark 9.55. Exercise 9.9 generalizes to 2π – periodic functions on $\mathbb{R}^d$, i.e. functions such that $f(\theta + 2\pi e_i) = f(\theta)$ for all $i = 1, 2, \ldots, d$ where $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$. A trigonometric polynomial $p(\theta)$ is a function of $\theta \in \mathbb{R}^d$ of the form

$$p(\theta) = \sum_{n \in \Gamma} \alpha_n e^{in\cdot\theta}$$

where Γ is a finite subset of $\mathbb{Z}^d$. The assertion is again that these trigonometric polynomials are dense in the 2π – periodic functions relative to the supremum norm.

Exercise 9.10. Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$, then $\mathbb{D} := \operatorname{span}\{e^{i\lambda\cdot x} : \lambda \in \mathbb{R}^d\}$ is a dense subspace of $L^p(\mu)$ for all $1 \le p < \infty$. **Hints:** By Corollary 9.8, $C_c(\mathbb{R}^d)$ is a dense subspace of $L^p(\mu)$. For $f \in C_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$, let

$$f_N(x) := \sum_{n \in \mathbb{Z}^d} f(x + 2\pi N n).$$

Show $f_N \in BC(\mathbb{R}^d)$ and $x \to f_N(Nx)$ is 2π – periodic, so by Exercise 9.9, $x \to f_N(Nx)$ can be approximated uniformly by trigonometric polynomials. Use this fact to conclude that $f_N \in \bar{\mathbb{D}}^{L^p(\mu)}$. After this show $f_N \to f$ in $L^p(\mu)$.

Exercise 9.11. Suppose that μ and ν are two finite measures on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\mu(x) = \int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\nu(x) \tag{9.28}$$

for all $\lambda \in \mathbb{R}^d$. Show $\mu = \nu$.

Hint: Perhaps the easiest way to do this is to use Exercise 9.10 with the measure μ being replaced by $\mu + \nu$. Alternatively, use the method of proof of Exercise 9.9 to show Eq. (9.28) implies $\int_{\mathbb{R}^d} f d\mu(x) = \int_{\mathbb{R}^d} f d\nu(x)$ for all $f \in C_c(\mathbb{R}^d)$.

Exercise 9.12. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$. Further assume there exists an $\epsilon > 0$ such that $C := \int_{\mathbb{R}^d} e^{\epsilon|x|} d\mu(x) < \infty$. Show the space $\mathcal{P}(\mathbb{R}^d)$ of polynomials on $\mathbb{R}^d$ are dense in $L^p(\mu)$ for all $1 \le p < \infty$. Here is a possible outline.

Outline: For $\lambda \in \mathbb{R}^d$ and $n \in \mathbb{N}$ let $f_n(x) = (\lambda \cdot x)^n / n!$

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-\epsilon t} = (\alpha/\epsilon)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/\epsilon)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-\epsilon|x|}\right) |\lambda|^{pn} e^{\epsilon|x|}$$
to find an estimate on $\|f_n\|_p$.
2. Use your estimate on $\|f_n\|_p$ to show there exists $\delta > 0$ such that $\sum_{n=0}^\infty \|f_n\|_p < \infty$ when $|\lambda| \le \delta$ and conclude for $|\lambda| \le \delta$ that $e^{i\lambda\cdot x} = L^p(\mu)\text{–}\sum_{n=0}^\infty f_n(x)$. From this it follows that $\int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\mu(x) = 0$ when $|\lambda| \le \delta$.
3. Let $\lambda \in \mathbb{R}^d$ ($|\lambda|$ not necessarily small) and set $g(t) := \int_{\mathbb{R}^d} e^{it\lambda\cdot x} d\mu(x)$ for $t \in \mathbb{R}$. Show $g \in C^\infty(\mathbb{R})$ and
$$g^{(n)}(t) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{it\lambda\cdot x} d\mu(x) \text{ for all } n \in \mathbb{N}.$$

4. Let $T = \sup\{\tau \geq 0 : g|_{[0,\tau]} \equiv 0\}$. By Step 2., $T \geq \delta$. If $T < \infty$, use Step 3. to conclude
$$\int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{iT\lambda \cdot x} d\mu(x) = 0 \text{ for all } n \in \mathbb{N}.$$
Then use Step 2. again to conclude
$$\int_{\mathbb{R}^d} e^{i(T+t)\lambda \cdot x} d\mu(x) = 0 \text{ for all } t \leq \delta / |\lambda|$$
which violates the definition of T and therefore $T = \infty$.
5. Now finish by appealing to Exercise 9.10.

Proof. The assertion that $\tau_z : L^p \to L^p$ is an isometric isomorphism follows from translation invariance of Lebesgue measure and the fact that $\tau_{-z} \circ \tau_z = id$. For the continuity assertion, observe that

$$\|\tau_z f - \tau_y f\|_p = \|\tau_{-y}(\tau_z f - \tau_y f)\|_p = \|\tau_{z-y} f - f\|_p$$

from which it follows that it is enough to show $\tau_z f \to f$ in L^p as $z \to 0 \in \mathbb{R}^n$.

When $f \in C_c(\mathbb{R}^n)$, $\tau_z f \to f$ uniformly and since the $K := \cup_{|z|\leq 1}\text{supp}(\tau_z f)$ is compact, it follows by the dominated convergence theorem that $\tau_z f \to f$ in L^p as $z \to 0 \in \mathbb{R}^n$. For general $g \in L^p$ and $f \in C_c(\mathbb{R}^n)$,

$$\|\tau_z g - g\|_p \leq \|\tau_z g - \tau_z f\|_p + \|\tau_z f - f\|_p + \|f - g\|_p = \|\tau_z f - f\|_p + 2\|f - g\|_p$$

and thus

$$\limsup_{z\to 0} \|\tau_z g - g\|_p \leq \limsup_{z\to 0} \|\tau_z f - f\|_p + 2\|f - g\|_p = 2\|f - g\|_p .$$

Because $C_c(\mathbb{R}^n)$ is dense in L^p, the term $\|f - g\|_p$ may be made as small as we please. ∎

Definition 9.14. Suppose that (X, τ) is a topological space and μ is a measure on $\mathcal{B}_X = \sigma(\tau)$. For a measurable function $f : X \to \mathbb{C}$ we define the essential support of f by

(9.5)
$$\text{supp}_\mu(f) = \{x \in U : \mu(\{y \in V : f(y) \neq 0\}) > 0 \text{ for all neighborhoods } V \text{ of } x\}.$$

It is not hard to show that if $\text{supp}(\mu) = X$ (see Definition 7.40) and $f \in C(X)$ then $\text{supp}_\mu(f) = \text{supp}(f) := \overline{\{f \neq 0\}}$, see Exercise 9.5.

Lemma 9.15. *Suppose (X, τ) is second countable and $f : X \to \mathbb{C}$ is a measurable function and μ is a measure on $\mathcal{B}_X$. Then $X := U \setminus \text{supp}_\mu(f)$ may be described as the largest open set such that $f 1_W(x) = 0$ for μ – a.e. x. Equivalently put, $C := \text{supp}_\mu(f)$ is the smallest closed subset of X such that $f = f 1_C$ a.e.*

Proof. To verify that the two descriptions of $\text{supp}_\mu(f)$ are equivalent, suppose $\text{supp}_\mu(f)$ is defined as in Eq. (9.5) and $W := X \setminus \text{supp}_\mu(f)$. Then

$$\begin{aligned} W &= \{x \in X : \mu(\{y \in V : f(y) \neq 0\}) = 0 \text{ for some neighborhood } V \text{ of } x\} \\ &= \cup \{V \subset_o X : \mu(f 1_V \neq 0) = 0\} \\ &= \cup \{V \subset_o X : f 1_V = 0 \text{ for } \mu - \text{a.e.}\} . \end{aligned}$$

So to finish the argument it suffices to show $\mu(f 1_W \neq 0) = 0$. To to this let $\mathcal{U}$ be a countable base for τ and set

$$\mathcal{U}_f := \{V \in \mathcal{U} : f 1_V = 0 \text{ a.e.}\}.$$

Then it is easily seen that $W = \cup \mathcal{U}_f$ and since $\mathcal{U}_f$ is countable $\mu(f 1_W \neq 0) \leq \sum_{V \in \mathcal{U}_f} \mu(f 1_V \neq 0) = 0$. ∎

Lemma 9.16. *Suppose $f, g, h : \mathbb{R}^n \to \mathbb{C}$ are measurable functions and assume that x is a point in $\mathbb{R}^n$ such that $|f| * |g|(x) < \infty$ and $|f| * (|g| * |h|)(x) < \infty$, then*

1. $f * g(x) = g * f(x)$
2. $f * (g * h)(x) = (f * g) * h(x)$
3. *If $z \in \mathbb{R}^n$ and $\tau_z(|f| * |g|)(x) = |f| * |g|(x - z) < \infty$, then*
$$\tau_z(f * g)(x) = \tau_z f * g(x) = f * \tau_z g(x)$$

4. *If* $x \notin \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$ *then* $f * g(x) = 0$ *and in particular,* $\operatorname{supp}_m(f * g) \subset \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$ *where in defining* $\operatorname{supp}_m(f * g)$ *we will use the convention that "*$f * g(x) \neq 0$*" when* $|f| * |g|(x) = \infty$*.*

Proof. For item 1.,

$$|f| * |g|(x) = \int_{\mathbb{R}^n} |f|(x-y)\,|g|(y)dy = \int_{\mathbb{R}^n} |f|(y)\,|g|(y-x)dy = |g| * |f|(x)$$

where in the second equality we made use of the fact that Lebesgue measure invariant under the transformation $y \to x - y$. Similar computations prove all of the remaining assertions of the first three items of the lemma.

Item 4. Since $f * g(x) = \tilde{f} * \tilde{g}(x)$ if $f = \tilde{f}$ and $g = \tilde{g}$ a.e. we may, by replacing f by $f1_{\operatorname{supp}_m(f)}$ and g by $g1_{\operatorname{supp}_m(g)}$ if necessary, assume that $\{f \neq 0\} \subset \operatorname{supp}_m(f)$ and $\{g \neq 0\} \subset \operatorname{supp}_m(g)$. So if $x \notin (\operatorname{supp}_m(f) + \operatorname{supp}_m(g))$ then $x \notin (\{f \neq 0\} + \{g \neq 0\})$ and for all $y \in \mathbb{R}^n$, either $x - y \notin \{f \neq 0\}$ or $y \notin \{g \neq 0\}$. That is to say either $x - y \in \{f = 0\}$ or $y \in \{g = 0\}$ and hence $f(x-y)g(y) = 0$ for all y and therefore $f * g(x) = 0$. This shows that $f * g = 0$ on $\mathbb{R}^n \setminus \left(\overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}\right)$ and therefore

$$\mathbb{R}^n \setminus \left(\overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}\right) \subset \mathbb{R}^n \setminus \operatorname{supp}_m(f * g),$$

i.e. $\operatorname{supp}_m(f * g) \subset \overline{\operatorname{supp}_m(f) + \operatorname{supp}_m(g)}$. ∎

Remark 9.17. Let A, B be closed sets of $\mathbb{R}^n$, it is not necessarily true that $A + B$ is still closed. For example, take

$$A = \{(x, y) : x > 0 \text{ and } y \geq 1/x\} \text{ and } B = \{(x, y) : x < 0 \text{ and } y \geq 1/|x|\},$$

then every point of $A + B$ has a positive y - component and hence is not zero. On the other hand, for $x > 0$ we have $(x, 1/x) + (-x, 1/x) = (0, 2/x) \in A + B$ for all x and hence $0 \in \overline{A + B}$ showing $A + B$ is not closed. Nevertheless if one of the sets A or B is compact, then $A + B$ is closed again. Indeed, if A is compact and $x_n = a_n + b_n \in A + B$ and $x_n \to x \in \mathbb{R}^n$, then by passing to a subsequence if necessary we may assume $\lim_{n\to\infty} a_n = a \in A$ exists. In this case

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (x_n - a_n) = x - a \in B$$

exists as well, showing $x = a + b \in A + B$.

Proposition 9.18. *Suppose that* $p, q \in [1, \infty]$ *and* p *and* q *are conjugate exponents,* $f \in L^p$ *and* $g \in L^q$*, then* $f * g \in BC(\mathbb{R}^n)$*,* $\|f * g\|_u \leq \|f\|_p \|g\|_q$ *and if* $p, q \in (1, \infty)$ *then* $f * g \in C_0(\mathbb{R}^n)$*.*

Proof. The existence of $f * g(x)$ and the estimate $|f * g|(x) \leq \|f\|_p \|g\|_q$ for all $x \in \mathbb{R}^n$ is a simple consequence of Holders inequality and the translation invariance of Lebesgue measure. In particular this shows $\|f * g\|_u \leq \|f\|_p \|g\|_q$. By relabeling p and q if necessary we may assume that $p \in [1, \infty)$. Since

$$\|\tau_z (f * g) - f * g\|_u = \|\tau_z f * g - f * g\|_u \leq \|\tau_z f - f\|_p \|g\|_q \to 0 \text{ as } z \to 0$$

it follows that $f * g$ is uniformly continuous. Finally if $p, q \in (1, \infty)$, we learn from Lemma 9.16 and what we have just proved that $f_m * g_m \in C_c(\mathbb{R}^n)$ where