56.11. **Solutions to Last Homework Assignment.**

6.18. Recall $\sigma(E) = n \cdot m(E_1)$ where $E_1 = \{rx : 0 < r \leq 1, x \in E\}$. For all rotations R, $(RE)_1 = RE_1$. Therefore

$$\begin{aligned}\sigma(RE) &= n\ m((RE)_1) = n\ m(RE_1)\\ &= n\ (\det R) m(E_1) = n\ m(E_1) = \sigma(E),\end{aligned}$$

showing σ is rotation invariant. ■

6.19.

$$\begin{aligned}\int_{\alpha \leq |x| \leq \beta} |x|^a \left|\ln |x|\right|^b dx &= c_n \int_\alpha^\beta r^a \left|\ln r\right|^b r^{n-1} dr\\ &= c_n \int_\alpha^\beta r^{a+n-1} \left|\ln r\right|^b dr\end{aligned}$$

where $c_n = \sigma(S^{n-1})$. Letting $u = \ln r$, i.e. $r = e^u$, we find

$$\int_{\alpha \leq |x| \leq \beta} |x|^a \left|\ln |x|\right|^b dx = c_n \int_{\ln \alpha}^{\ln \beta} e^{(a+n)u} \left|u\right|^b du.$$

Therefore,

$$\int_{|x| \leq 1/2} |x|^a \left|\ln |x|\right|^b dx = c_n \int_{-\infty}^{-\ln 2} e^{(a+n)u} \left|u\right|^b du$$

is finite iff either $(a + n) > 0$ or $a = -n$ and $b < -1$. Similarly,

$$\int_{2 \leq |x|} |x|^a \left|\ln |x|\right|^b dx = c_n \int_{\ln 2}^{\infty} e^{(a+n)u} \left|u\right|^b du$$

which is finite iff $a + n < 0$ or $a = -n$ and $b < -1$. ■

7.3. Let $W := X \setminus \mathrm{supp}(\nu)$, then $x \in W$ iff there exists an open neighborhood V of x such that $\nu(V) = 0$. Now clearly $V \subset W$ and therefore W may be written as a union of open sets and hence W is open as well. This proves (1).

(2) Suppose that W' is an open set such that $\nu(W') = 0$, then clearly $W' \subset W$ by definition. So to finish the proof it suffices to show $\nu(W) = 0$. Let D and $\mathcal{V}$ be as in the hint and suppose that $x \in W$ and $\epsilon > 0$ is such that $\nu(B_x(\epsilon)) = 0$. Let $n \in \mathbb{N}$ be chosen so that $\frac{2}{n} < \epsilon$ and choose $y \in D$ such that $d(x, y) < 1/n$. Then $x \in B_y(1/n) \subset B_x(\epsilon)$ and hence $\nu(B_y(1/n)) = 0$ showing

$$W := \cup\{V \in \mathcal{V} : \nu(V) = 0\}.$$

This show W is a countable union of null sets and therefore W is a null set as well, i.e. $\nu(W) = 0$. ■

7.4.

(1) $\lambda \in \mathrm{supp}(\nu)$ iff for all $\epsilon > 0$,

$$0 < \nu(B_\lambda(\epsilon)) = \mu(f^{-1}(B_\lambda(\epsilon))) = \mu(|f - \lambda| < \epsilon) > 0.$$

This clearly proves $\mathrm{essran}(f) = \mathrm{supp}(\nu)$.

(2) Item (1) and the results of Exercise 7.3 prove (2) since

$$\mu(f \notin \mathrm{essran}(f)) = \mu(f \notin \mathrm{supp}(\nu)) = \nu(\mathbb{C} \setminus \mathrm{supp}(\nu)) = 0.$$

(3) If $\lambda \notin F$, then there exists $\epsilon > 0$ such that $B_\lambda(\epsilon) \subset F^c$ and therefore

$$\mu(|f - \lambda| < \epsilon) = \mu\left(f^{-1}(B_\lambda(\epsilon))\right) \leq \mu(f \notin F) = 0.$$

This shows $\lambda \notin C$, so $F^c \subset C^c$ or $C \subset F$.

(4) Let $M = \sup\{|\lambda| : \lambda \in \text{essran}(f)\}$, then clearly $\text{essran}(f) \subset C_0(M)$ so that $f \in C_0(M)$ a.e., i.e. $|f| \le M$ a.e. So by Remark 7.4, $\|f\|_\infty \le M$. Conversely, it is also clear that $\text{essran}(f) \subset C_0(\|f\|_\infty)$ and hence $M \le \|f\|_\infty$.

■

7.9. Notice that making the change of variables $u = \ln x$ below gives

$$\int_\alpha^\beta x^{-a} |\log x|^b \, dx = \int_{\ln \alpha}^{\ln \beta} e^{-au} |u|^b e^u du.$$

so that

$$\int_0^{1/2} x^{-a} |\log x|^b \, dx = \int_{-\infty}^{-\ln 2} e^{(1-a)u} |u|^b \, du$$

which is finite iff $a < 1$ or $a = 1$ and $b < -1$ (so if $b < -1$ we get a finite answer iff $a \le 1$),

$$\int_2^\infty x^{-a} |\log x|^b \, dx = \int_{\ln 2}^\infty e^{(1-a)u} |u|^b \, du < \infty$$

which is finite iff $a > 1$ or $a = 1$ and $b < -1$ (so if $b < -1$ we get a finite answer iff $a \ge 1$),

$$\int_0^1 x^{-a} |\log x|^b \, dx = \int_{-\infty}^0 e^{(1-a)u} |u|^b \, du$$

which is finite iff $a < 1$ and $b > -1$ (so if $b > -1$ we get a finite answer iff $a < 1$) and

$$\int_1^\infty x^{-a} |\log x|^b \, dx = \int_0^\infty e^{(1-a)u} |u|^b \, du < \infty$$

which is finite iff $a > 1$ and $b > -1$ (so if $b > -1$ we get a finite answer iff $a > 1$).

I will use these facts below The first two with $b = -2$ and the second two with $b = 0$..

a) For $p_0 < p < p_1 < \infty$, let $A(x) = 1_{x \ge 1} \frac{1}{x}$ and $B(x) = 1_{x \le 1} \frac{1}{x}$ then $\int |A|^p < \infty$ iff $p > 1$ and $\int |B|^p < \infty$ iff $p < 1$. Therefore, letting $f = A^{1/p_0} + B^{1/p_1}$ we have

$$\|f\|_p^p = \int A^{p/p_0} + \int B^{p/p_1} < \infty$$

iff $p > p_0$ and $p < p_1$. If $p_1 = \infty$ take $f(x) = A^{1/p_0}(x) + \left|\frac{1}{\log x}\right| 1_{x<1}$, $\|f\|_\infty = \infty$ while

$$\|f\|_p^p = \int A^{p/p_0} + \int \left|\frac{1}{\log x}\right|^p < \infty$$

for all $p_0 < p < \infty$.

b) $p_0 \le p \le p_1 < \infty$. Let $A(x) = 1_{x \ge 2} \frac{1}{x(\ln x)^2}$ and $B(x) = 1_{x \le 1/2} \frac{1}{x(\ln x)^2}$ then $\int |A|^p < \infty$ iff $p \ge 1$ and $\int |B|^p < \infty$ iff $p \le 1$. Therefore, letting $f = A^{1/p_0} + B^{1/p_1}$ we have

$$\|f\|_p^p = \int A^{p/p_0} + \int B^{p/p_1} < \infty$$

iff $p \ge p_0$ and $p \le p_1$. If $p_1 = \infty$, let $f = A^{1/p_0}$.

c) If $p < \infty$, let A and B be as b) and set $f(x) = A^{1/p} + B^{1/p}$. If $p = \infty$, let $f(x) = 1$. ■

7.12. By Hölder's inequality,

$$\int |k(x,y)f(y)|\, d\nu(y) \leq \left(\int |k(x,y)|^2\, d\nu(y)\right)^{1/2} \|f\|_{L^2(\nu)}$$

and by Tonelli's theorem

$$\int_X d\mu(x) \int |k(x,y)|^2\, d\nu(y) = \|k\|^2_{L^2(\mu\otimes\nu)}.$$

This shows $\int |k(x,y)f(y)|\, d\nu(y) < \infty$ for $\mu -$ a.e. x. Since

$$|Kf(x)|^2 \leq \left[\int |k(x,y)f(y)|\, d\nu(y)\right]^2 \leq \int |k(x,y)|^2\, d\nu(y)\; \|f\|^2_{L^2(\nu)}$$

it again follows by Tonelli's theorem that

$$\|Kf\|^2_{L^2(\mu)} \leq \|k\|^2_{L^2(\mu\otimes\nu)} \|f\|^2_{L^2(\nu)}.$$

This finishes the argument since it is easily verified that K is linear. ■