Math 240C Homework Problem List for S2018

0.1 Homework C1. Due Friday, April 6, 2018

- **Hand in**: 1.3, 1.4, 1.5, 1.13, 1.14
- **Look at**: 1.12

0.2 Homework C2. Due Friday, April 13, 2018 (L^p inequalities)

- **Hand in**: 1.1, 1.2, 1.6, 1.7, 1.8, 1.9
- **Look at**: 1.10, 1.11

Please note that Exercise 1.6 has been corrected.

Problems to Solve

Exercise 1.1. If (X, ρ) is a metric space and μ is a **finite** measure on $(X, \mathcal{B}_X)$, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set F and open set V such that $F \subset A \subset V$ and $\mu(V \setminus F) = \mu(F \triangle V) < \varepsilon$.

You may find information in the supplement helpful for this problem. Here are some more suggestions.

1. Let $\mathcal{B}_0$ denote those $A \subset X$ such that for all $\varepsilon > 0$ there exists a closed set F and open set V such that $F \subset A \subset V$ and $d_\mu(F, V) = \mu(V \setminus F) < \varepsilon$.
2. Show $\mathcal{B}_0$ contains all closed (or open if you like).
3. Show $\mathcal{B}_0$ is a σ-algebra.
4. Explain why this proves the result.

Exercise 1.2. Let (X, ρ) be a metric space and μ be a measure on $(X, \mathcal{B}_X)$. If there exists open sets, $\{V_n\}_{n=1}^\infty$, of X such that $V_n \uparrow X$ and $\mu(V_n) < \infty$ for all n, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set F and open set V such that $F \subset A \subset V$ and $d_\mu(F, V) = \mu(V \setminus F) < \varepsilon$. **Hints:**

1. Show it suffices to prove; for all $\varepsilon > 0$ and $A \in \mathcal{B}_X$, there exists an open set $V \subset X$ such that $A \subset V$ and $\mu(V \setminus A) < \varepsilon$.
2. Now you must verify the assertion above holds. For this, you may find it useful to apply Exercise 1.1 to the measures, $\mu_n : \mathcal{B}_X \to [0, \mu(V_n)]$, defined by $\mu_n(A) := \mu(A \cap V_n)$ for all $A \in \mathcal{B}_X$. The ε in Exercise 1.1 should be replaced by judiciously chosen small quantities depending on n.

Exercise 1.3 (Folland Problem 2.62 on p. 80.). Rotation invariance of surface measure on S^{n-1}.

Exercise 1.4 (Folland Problem 2.64 on p. 80.). On the integrability of $|x|^a \left|\log|x|\right|^b$ for x near 0 and x near ∞ in $\mathbb{R}^n$.

Exercise 1.5. Show, using Problem 1.3 that

$$\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega) = \frac{1}{d}\delta_{ij}\sigma\left(S^{d-1}\right).$$

Hint: show $\int_{S^{d-1}} \omega_i^2 d\sigma(\omega)$ is independent of i and therefore

$$\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d}\sum_{j=1}^d \int_{S^{d-1}} \omega_j^2 d\sigma(\omega).$$

Exercise 1.6 (Folland 6.38 on p. 199.). Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $f : X \to \mathbb{C}$ is a measurable function, $0 < p < \infty$, $\lambda_f(\alpha) := \mu(|f| > \alpha)$ for all $\alpha \in (0, \infty)$, and

$$M_p(f) := \sum_{k=-\infty}^{\infty} 2^{kp}\lambda_f\left(2^k\right).$$

Show

$$\left(1 - 2^{-p}\right) M_p(f) \le \int_X |f|^p d\mu \le 2^p M_p(f) \tag{1.1}$$

which then implies $f \in L^p(\mu)$ iff $M_p(f) < \infty$.

Hint: first note that

$$\int_X |f|^p d\mu = \sum_{k\in\mathbb{Z}} \int_{\{2^k < |f| \le 2^{k+1}\}} |f|^p d\mu. \tag{1.2}$$

Exercise 1.7 (Folland 6.39 on p. 199.). Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $f : X \to \mathbb{C}$ is a measurable function, $0 < p < \infty$, and $f \in L^p(\mu)$. Show

$$\lim_{\alpha\to 0} \alpha^p \lambda_f(\alpha) = 0 = \lim_{\alpha\to\infty} \alpha^p \lambda_f(\alpha).$$

Hint: for the limit, $\lim_{\alpha\to 0} \alpha^p \lambda_f(\alpha) = 0$, start with the special case where f is a simple function.

Exercise 1.8 (Folland 6.27 on p. 196. Hilbert's Inequality). Hint: See Theorem **??** which is Theorem 6.20 in Folland .

Exercise 1.9 (Folland 6.22). Exercise, Folland 6.22 on p. 192.

Exercise 1.10 (Global Integration by Parts Formula). Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are locally absolutely continuous functions[1] such that $f'g$, fg', and fg are all Lebesgue integrable functions on $\mathbb{R}$. Prove the following integration by parts formula;

$$\int_{\mathbb{R}} f'(x) \cdot g(x)\, dx = -\int_{\mathbb{R}} f(x) \cdot g'(x)\, dx. \tag{1.3}$$

[1] This means that f and g restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.

Similarly show that; if $f, g : [0,\infty) \to [0,\infty)$ are locally absolutely continuous functions such that $f'g$, fg', and fg are all Lebesgue integrable functions on $[0,\infty)$, then

$$\int_0^\infty f'(x) \cdot g(x)\, dx = -f(0)\, g(0) - \int_0^\infty f(x) \cdot g'(x)\, dx. \tag{1.4}$$

Outline: 1. First use the theory developed to see that Eq. (1.3) holds if $f(x) = 0$ for $|x| \geq N$ for some $N < \infty$.

2. Let $\psi : \mathbb{R} \to [0,1]$ be a continuously differentiable function such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$.[2] For any $\varepsilon > 0$ let $\psi_\varepsilon(x) = \psi(\varepsilon x)$ Write out the identity in Eq. (1.3) with $f(x)$ being replaced by $f(x)\, \psi_\varepsilon(x)$.

3. Now use the dominated convergence theorem to pass to the limit as $\varepsilon \downarrow 0$ in the identity you found in step 2.

4. A similar outline works to prove Eq. (1.4).

Exercise 1.11 (Heisenberg's Inequality). Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a locally absolutely continuous function[3], show

$$\int_{\mathbb{R}} |f(x)|^2\, dx \leq 2 \left[\int_{\mathbb{R}} |x f(x)|^2\, dx\right]^{1/2} \left[\int_{\mathbb{R}} |f'(x)|^2\, dx\right]^{1/2}. \tag{1.5}$$

Hint: assuming the right hand side of the above inequality is finite show

$$\int_{\mathbb{R}} |f(x)|^2\, dx = -2\, \mathrm{Re} \int_{\mathbb{R}} x \bar{f}(x)\, f'(x)\, dx. \tag{1.6}$$

Exercise 1.12. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0,1])$. **Hints:** you might start by first showing $\lim_{t \downarrow 0} f^{(n)}(t) = 0$ for all $n \in \mathbb{N}_0$.

Exercise 1.13. If $f \in L^1_{loc}(\mathbb{R}^d, m)$ and $\varphi \in C_c^1(\mathbb{R}^d)$, then $f * \varphi \in C^1(\mathbb{R}^d)$ and $\partial_i (f * \varphi) = f * \partial_i \varphi$. Moreover if $\varphi \in C_c^\infty(\mathbb{R}^d)$ then $f * \varphi \in C^\infty(\mathbb{R}^d)$.

Exercise 1.14 (Integration by Parts). Suppose that $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to f(x,y) \in \mathbb{C}$ and $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to g(x,y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^d$, $x \to f(x,y)$ and $x \to g(x,y)$ are continuously differentiable. Also assume $f \cdot g$, $\partial_x f \cdot g$ and $f \cdot \partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R} \times \mathbb{R}^{d-1}$, where $\partial_x f(x,y) := \frac{d}{dt} f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R} \times \mathbb{R}^{d-1}} \partial_x f(x,y) \cdot g(x,y) dx dy = -\int_{\mathbb{R} \times \mathbb{R}^{d-1}} f(x,y) \cdot \partial_x g(x,y) dx dy. \tag{1.7}$$

(Note: this result and Fubini's theorem proves Lemma **??**.)

Hints: Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0 \in \mathbb{R}$ and set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. First verify Eq. (1.7) with $f(x,y)$ replaced by $\psi_\varepsilon(x)\, f(x,y)$ by doing the $x-$ integral first. Then use the dominated convergence theorem to prove Eq. (1.7) by passing to the limit, $\varepsilon \downarrow 0$.

[2] You may assume the existence of such a ψ, we will deal with this later.

[3] This means that f restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.