Math 240C Homework Problem List for S2018

0.1 Homework C1. Due Friday, April 6, 2018

- Hand in: $1.3, 1.4, 1.5, 1.25, 1.26$
- Look at: $1.23$

0.2 Homework C2. Due Friday, April 13, 2018 ($L^p$ inequalities)

- Hand in: $1.1, 1.2, 1.6, 1.7, 1.19, 1.20$
- Look at: $1.21, 1.22$

0.3 Homework C3. Due Friday, April 20, 2018 (Fourier Series problems)

- Hand in: $1.8, 1.10, 1.11, 1.12, 1.13, 1.14, 1.16$
- Look at: $1.24$ (done in class), $1.9, 1.15$

0.4 Homework C4. Due Friday, April 27, 2018 (Fourier Transform problems)

- Hand in: $1.27, 1.29, 1.30, 1.31, 1.34, 1.35$
- Look at: $1.28$

0.5 Homework C5. Due Friday, May 4, 2018

- Hand in: $1.32, 1.33, 1.35, 1.36$
Problems to Solve

Exercise 1.1. If \((X, \rho)\) is a metric space and \(\mu\) is a finite measure on \((X, \mathcal{B}_X)\), then for all \(A \in \mathcal{B}_X\) and \(\varepsilon > 0\) there exists a closed set \(F\) and open set \(V\) such that \(F \subset A \subset V\) and \(\mu(V \setminus F) = \mu(F \triangle V) < \varepsilon\).

You may find information in the supplement helpful for this problem. Here are some more suggestions.

1. Let \(B_0\) denote those \(A \subset X\) such that for all \(\varepsilon > 0\) there exists a closed set \(\overline{F}\) and open set \(\overline{V}\) such that \(\overline{F} \subset A \subset \overline{V}\) and \(\mu\left(\overline{V} \setminus \overline{F}\right) = \mu\left(\overline{F} \triangle \overline{V}\right) < \varepsilon\).
2. Show \(B_0\) contains all closed (or open if you like).
3. Show \(B_0\) is a \(\sigma\)-algebra.
4. Explain why this proves the result.

Exercise 1.2. Let \((X, \rho)\) be a metric space and \(\mu\) be a measure on \((X, \mathcal{B}_X)\). If there exists open sets, \(\{V_n\}_{n=1}^\infty\), of \(X\) such that \(V_n \uparrow X\) and \(\mu(V_n) < \infty\) for all \(n\), then for all \(A \in \mathcal{B}_X\) and \(\varepsilon > 0\) there exists a closed set \(F\) and open set \(V\) such that \(F \subset A \subset V\) and \(\mu(V \setminus F) = \mu(F \triangle V) < \varepsilon\). Hints:

1. Show it suffices to prove; for all \(\varepsilon > 0\) and \(A \in \mathcal{B}_X\), there exists an open set \(V \subset X\) such that \(A \subset V\) and \(\mu(V \setminus A) < \varepsilon\).
2. Now you must verify the assertion above holds. For this, you may find it useful to apply Exercise 1.1 to the measures, \(\mu_n : \mathcal{B}_X \to [0, \mu(V_n)]\), defined by \(\mu_n(A) := \mu(A \cap V_n)\) for all \(A \in \mathcal{B}_X\). The \(\varepsilon\) in Exercise 1.1 should be replaced by judiciously chosen small quantities depending on \(n\).

Exercise 1.3 (Folland Problem 2.62 on p. 80.). Rotation invariance of surface measure on \(S^{n-1}\).

Exercise 1.4 (Folland Problem 2.64 on p. 80.). On the integrability of \(|x|^a \log |x|^b\) for \(x\) near \(0\) and \(x\) near \(\infty\) in \(\mathbb{R}^n\).

Exercise 1.5. Show, using Problem L3 that

\[
\int_{S^{d-1}} \omega_i \sigma_j d\sigma(\omega) = \frac{1}{d} \delta_{ij} \sigma(S^{d-1}).
\]

Hint: show \(\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega)\) is independent of \(i\) and therefore

\[
\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d} \sum_{j=1}^d \int_{S^{d-1}} \omega_j^2 d\sigma(\omega).
\]

Exercise 1.6 (Folland 6.38 on p. 199.). Suppose \((X, \mathcal{M}, \mu)\) is a measure space, \(f : X \to \mathbb{C}\) is a measurable function, \(0 < p < \infty\), \(\lambda_f(\alpha) := \mu(|f| > \alpha)\) for all \(\alpha \in (0, \infty)\), and

\[
M_p(f) := \sum_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k).
\]

Show

\[
(1 - 2^{-p}) M_p(f) \leq \int_X |f|^p d\mu \leq 2^p M_p(f) \tag{1.1}
\]

which then implies \(f \in L^p(\mu)\) iff \(M_p(f) < \infty\). Hints: first note that

\[
\int_X |f|^p d\mu = \sum_{k \in \mathbb{Z}} \int_{\{2^k < |f| \leq 2^{k+1}\}} |f|^p d\mu. \tag{1.2}
\]

Exercise 1.7 (Folland 6.39 on p. 199.). Suppose \((X, \mathcal{M}, \mu)\) is a measure space, \(f : X \to \mathbb{C}\) is a measurable function, \(0 < p < \infty\), and \(f \in L^p(\mu)\). Show

\[
\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0 = \lim_{\alpha \to \infty} \alpha^p \lambda_f(\alpha).
\]

Hint: for the limit, \(\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0\), start with the special case where \(f\) is a simple function.

Exercise 1.8. Show \(\sum_{k=1}^\infty k^{-2} = \pi^2/6\), by taking \(f(x) = x\) on \([-\pi, \pi]\) and computing \(\|f\|_2^2\) directly and then in terms of the Fourier Coefficients \(\tilde{f}\) of \(f\).

Exercise 1.9 (Riemann Lebesgue Lemma for Fourier Series). Show for \(f \in L^1\left([-\pi, \pi]^d\right)\) that \(\tilde{f} \in c_0(\mathbb{Z}^d)\), i.e. \(\tilde{f} : \mathbb{Z}^d \to \mathbb{C}\) and \(\lim_{k \to \infty} \tilde{f}(k) = 0\). Hint: if \(f \in L^2\left([-\pi, \pi]^d\right)\), this follows from Bessel’s inequality. Now use a density argument.

Exercise 1.10. Suppose \(f \in L^1([-\pi, \pi]^d)\) is a function such that \(\tilde{f} \in \ell^1(\mathbb{Z}^d)\) and set

\[
g(x) := \sum_{k \in \mathbb{Z}^d} \tilde{f}(k)e^{ik \cdot x} \text{ (pointwise)}.
\]
1. Problems to Solve

1. Show $g \in C_{per}(\mathbb{R}^d)$. 

2. Show $g(x) = f(x)$ for $m$ a.e. $x$ in $[-\pi,\pi]^d$. **Hint:** Show $\hat{g}(k) = \hat{f}(k)$ and apply Exercise 1.11 or results proved in class.

3. Conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$ and in particular $f \in L^p([-\pi,\pi]^d)$ for all $p \in [1,\infty]$.

**Exercise 1.11 (Smoothness implies decay).** We use the following notation.

**Notation:** Given a multi-index $\alpha \in \mathbb{Z}_+^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$, 

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \quad \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$ 

Further for $k \in \mathbb{N}_0$, let $f \in C^k_{per}(\mathbb{R}^d)$ iff $f \in C^k(\mathbb{R}^d) \cap C_{per}(\mathbb{R}^d)$, $\partial_x^\alpha f(x)$ exists and is continuous for $|\alpha| \leq k$.

Suppose $m \in \mathbb{N}_0$, $\alpha$ is a multi-index such that $|\alpha| \leq 2m$ and $f \in C_{per}^{2m}(\mathbb{R}^d)$. 

1. Using integration by parts, show (using Notation ??) that

$$(ik)^\alpha \hat{f}(k) = (\partial_x^\alpha f)|_{\phi_k}$$

for all $k \in \mathbb{Z}_+^d$.

Note: This equality implies

$$|\hat{f}(k)| \leq \frac{1}{k^{|\alpha|}} \|\partial_x^\alpha f\|_H \leq \frac{1}{k^{|\alpha|}} \|\partial_x^\alpha f\|_\infty.$$ 

2. Now let $\Delta f = \sum_{i=1}^d \partial_x^2 f/\partial x_i^2$, Working as in part 1) show

$$(1 - \Delta)^m f|_{\phi_k} = (1 + \|k\|^2)^m \hat{f}(k).$$

where $\|k\|^2 = \sum_{j=1}^d k_j^2$.

**Exercise 1.12 (A Sobolev Imbedding Theorem).** Suppose $s \in \mathbb{R}$ and $\{c_k \in \mathbb{C} : k \in \mathbb{Z}_+^d\}$ are coefficients such that

$$\sum_{k \in \mathbb{Z}_+^d} |c_k|^2 (1 + |k|^2)^s < \infty.$$ 

Show if $s > \frac{d}{2} + m$, the function $f$ defined by

$$f(x) = \sum_{k \in \mathbb{Z}_+^d} c_k e^{ik \cdot x}$$

is in $C_{per}^m(\mathbb{R}^d)$. **Hint:** Work as in the above remark to show

$$\sum_{k \in \mathbb{Z}_+^d} |c_k| |k|^\alpha < \infty \text{ for all } |\alpha| \leq m.$$ 

**Exercise 1.13 (Poisson Summation Formula).** Let $F \in L^1(\mathbb{R}^d)$,

$$E := \left\{ x \in \mathbb{R}^d : \sum_{k \in \mathbb{Z}_+^d} |F(x + 2\pi k)| = \infty \right\}$$

and set (see Section ?? below for an introduction to the Fourier transform)

$$\hat{F}(k) := \langle 2\pi \rangle^{-d/2} \int_{\mathbb{R}^d} F(x) e^{-ik \cdot x} \, dx \text{ for } k \in \mathbb{Z}_+^d.$$ 

Further assume $\hat{F} \in \ell^1(\mathbb{Z}_+^d)$. [This can be achieved by assuming $F$ is sufficiently differentiable with the derivatives being integrable like in Exercise 1.11]

1. Show $m(E) = 0$ and $E + 2\pi k = E$ for all $k \in \mathbb{Z}_+^d$. **Hint:** Compute

$$\int_{[-\pi,\pi]^d} |F(x + 2\pi k)| \, dx.$$ 

2. Let

$$f(x) := \left\{ \begin{array}{ll}
\sum_{k \in \mathbb{Z}_+^d} F(x + 2\pi k) & \text{for } x \notin E \\
0 & \text{if } x \in E.
\end{array} \right.$$ 

Show $f \in L^1([-\pi,\pi]^d)$ and $\hat{f}(k) = (2\pi)^{-d/2} \hat{F}(k)$.

3. Using item 2) and the assumptions on $F$, show

$$f(x) = \sum_{k \in \mathbb{Z}_+^d} \hat{f}(k) e^{ik \cdot x} = \sum_{k \in \mathbb{Z}_+^d} (2\pi)^{-d/2} \hat{F}(k) e^{ik \cdot x} \text{ for } m - \text{a.e. } x,$$

i.e.

$$\sum_{k \in \mathbb{Z}_+^d} F(x + 2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}_+^d} \hat{F}(k) e^{ik \cdot x} \text{ for } m - \text{a.e. } x$$ 

and form this conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$.

**Hint:** see the hint for item 2) of Exercise 1.10

4. Suppose we now assume that $F \in C(\mathbb{R}^d)$ and $F$ satisfies $|F(x)| \leq C(1 + |x|)^{-s}$ for some $s > d$ and $C < \infty$. Under these added assumptions on $F$, show Eq. (1.4) holds for all $x \in \mathbb{R}^d$ and in particular

$$\sum_{k \in \mathbb{Z}_+^d} F(2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}_+^d} \hat{F}(k).$$
Exercise 1.14 (Heat Equation 1.). Let $[0, \infty) \times \mathbb{R} \ni (t, x) \mapsto u(t, x)$ be a continuous function such that $u(t, \cdot) \in C_{\text{per}}(\mathbb{R})$ for all $t \geq 0$, $\dot{u} := u_t$, $u_{xx}$, and $u_{xx}$ exist and are continuous when $t > 0$. Further assume that $u$ satisfies the heat equation $\dot{u} = \frac{1}{2}u_{xx}$. Let $\tilde{u}(t, k) := \langle u(t, \cdot) \vert \varphi_k \rangle$ for $k \in \mathbb{Z}$. Show for $t > 0$ and $k \in \mathbb{Z}$ that $\tilde{u}(t, k)$ is differentiable in $t$ and $\frac{d}{dt} \tilde{u}(t, k) = -k^2 \tilde{u}(t, k)/2$. Use this result to show

$$u(t, x) = \sum_{k \in \mathbb{Z}} e^{-\frac{k^2}{4}t} \tilde{f}(k) e^{ikx}$$  \hspace{1cm} (1.5)

where $f(x) := u(0, x)$ and as above

$$\tilde{f}(k) = \langle f \vert \varphi_k \rangle = \int_0^\pi f(y) e^{-iky} dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iky} dm(y).$$

Notice from Eq. (1.5) that $(t, x) \mapsto u(t, x)$ is $C^\infty$ for $t > 0$.

Exercise 1.15 (Heat Equation 2.). Let $q_t(x) := \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-\frac{k^2}{4}t} e^{ikx}$. Show:

1. Eq. (1.5) may be rewritten as

$$u(t, x) = \int_{-\pi}^{\pi} q_t(x - y) f(y) dy$$

and

2. $q_t(x)$ may be expressed as

$$q_t(x) = \sum_{k \in \mathbb{Z}} p_t(x + k2\pi)$$

where $p_t(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4t}}$.

3. Also show $u(t, x)$ may be written as

$$u(t, x) = (p_t \ast f)(x) := \int_{\mathbb{R}^d} p_t(x - y) f(y) dy.$$

Hint: To show $q_t(x) = \sum_{k \in \mathbb{Z}} p_t(x + k2\pi)$, use the Poisson summation formula (Exercise 1.13) and the Gaussian integration identity,

$$p_t(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} p_t(x) e^{ix\omega} dx = \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4t}}.$$  \hspace{1cm} (1.6)

Exercise 1.16 (Wave Equation). Let $u \in C^2(\mathbb{R} \times \mathbb{R})$ be such that $u(t, \cdot) \in C_{\text{per}}(\mathbb{R})$ for all $t \in \mathbb{R}$. Further assume that $u$ solves the wave equation, $u_{tt} = u_{xx}$. Let $f(x) := u(0, x)$ and $g(x) := \dot{u}(0, x)$. Show $\tilde{u}(t, k) := \langle u(t, \cdot), \varphi_k \rangle$ for $k \in \mathbb{Z}$ is twice continuously differentiable in $t$ and $\frac{d^2}{dt^2} \tilde{u}(t, k) = -k^2 \tilde{u}(t, k)$. Use this result to show

$$u(t, x) = \sum_{k \in \mathbb{Z}} \left( \hat{f}(k) \cos(kt) \frac{\sin kt}{k} e^{ikx} \right)$$  \hspace{1cm} (1.7)

with the sum converging absolutely. Also show that $u(t, x)$ may be written as

$$u(t, x) = \frac{1}{2} \left[ f(x + t) + f(x - t) \right] + \frac{1}{2} \int_{-t}^t g(x + \tau) d\tau.$$  \hspace{1cm} (1.8)

Hint: To show Eq. (1.7) implies (1.8) use

$$\cos kt = \frac{e^{ikt} + e^{-ikt}}{2},$$

$$\sin kt = \frac{e^{ikt} - e^{-ikt}}{2i},$$

and

$$\frac{e^{ikt} - e^{ik(t-x)}}{ik} = \int_{-t}^t e^{ik(x + \tau)} d\tau.$$

Exercise 1.17. Let $f \in L^1((-\pi, \pi))$ which we extend to a $2\pi$-periodic function on $\mathbb{R}$ and continue to denote by $f$. If there exists $q \in \mathbb{N}$ such that $f \left( x + \frac{2\pi}{q} \right) = f(x)$ for $m$-a.e. $x$, then $\hat{f}(k) = 0$ unless $q$ divides $k$.

Exercise 1.18. In this problem we assume the notation from subsection ?? with $d = 1$. For simplicity of notation we identify $L^2((-\pi, \pi), d\theta)$ with $2\pi$-periodic functions on $\mathbb{R}$ via,

$$L^2((-\pi, \pi), d\theta) \ni f \leftrightarrow \sum_{n \in \mathbb{Z}} f(x + n2\pi) 1_{(-\pi, \pi)}(x + n2\pi) \in L^2_{\text{per}}(\mathbb{R}).$$

Given $\alpha \in \mathbb{R}$ let $(U_\alpha f)(\theta) = f(\theta + \alpha 2\pi)$ wherein we have used the above identification. If $\alpha \notin \mathbb{Q}$ show

$$M_\alpha = \text{Nul} (U_\alpha - I) = \mathbb{C} \cdot 1.$$ 

If $\alpha \in \mathbb{Q}$ write $\alpha = \frac{p}{q}$ where $\gcd(p, q) = 1$, i.e. $p$ and $q$ are relatively prime. In this case show $M_\alpha = \text{Nul} (U_\alpha - I)$ consists of those $f \in L^2_{\text{per}}(\mathbb{R})$ such that $f \left( x + \frac{2\pi}{q} \right) = f(x)$ for $m$--a.e. $x$. [Consequently, combining this exercise with Mean Ergodic Theorem ?? shows,

$$\frac{1}{n} \sum_{k=0}^{n-1} U_{\alpha k} \xrightarrow{P_{M_\alpha}} \frac{1}{n}$$

where $M_\alpha$ depends on $\alpha$ as described above.]
Exercise 1.19 (Folland 6.27 on p. 196. Hilbert’s Inequality). Hint: See Theorem ?? which is Theorem 6.20 in Folland.


Exercise 1.21 (Global Integration by Parts Formula). Suppose that \( f, g: \mathbb{R} \to \mathbb{C} \) are locally absolutely continuous functions such that \( f', g, f g' \), and \( fg \) are all Lebesgue integrable functions on \( \mathbb{R} \). Prove the following integration by parts formula:

\[
\int_{\mathbb{R}} f'(x) \cdot g(x) \, dx = -\int_{\mathbb{R}} f(x) \cdot g'(x) \, dx. \tag{1.9}
\]

Similarly show that, if \( f, g: [0, \infty) \to [0, \infty) \) are locally absolutely continuous functions such that \( f', g, f g' \), and \( fg \) are all Lebesgue integrable functions on \([0, \infty)\), then

\[
\int_{0}^{\infty} f'(x) \cdot g(x) \, dx = -f(0)g(0) - \int_{0}^{\infty} f(x) \cdot g'(x) \, dx. \tag{1.10}
\]

Outline: 1. First use the theory developed to see that Eq. (1.9) holds if \( f(x) = 0 \) for \( |x| \geq N \) for some \( N < \infty \).

2. Let \( \psi: \mathbb{R} \to [0, 1] \) be a continuously differentiable function such that \( \psi(x) = 1 \) if \( |x| \leq 1 \) and \( \psi(x) = 0 \) if \( |x| \geq 2 \). For any \( \varepsilon > 0 \) let \( \psi_\varepsilon(x) = \varepsilon \psi(\varepsilon x) \)

Write out the identity in Eq. (1.9) with \( f(x) \) being replaced by \( f(x) \psi_\varepsilon(x) \).

3. Now use the dominated convergence theorem to pass to the limit as \( \varepsilon \downarrow 0 \) in the identity you found in step 2.

4. A similar outline works to prove Eq. (1.10).

Exercise 1.22 (Heisenberg’s Inequality). Suppose that \( f: \mathbb{R} \to \mathbb{C} \) is a locally absolutely continuous function such that

\[
\int_{\mathbb{R}} |f(x)|^2 \, dx \leq 2 \left[ \int_{\mathbb{R}} |xf(x)|^2 \, dx \right]^{1/2} \left[ \int_{\mathbb{R}} |f'(x)|^2 \, dx \right]^{1/2}. \tag{1.11}
\]

Hint: assuming the right hand side of the above inequality is finite show

\[
\int_{\mathbb{R}} |f(x)|^2 \, dx = -2 \text{Re} \int_{\mathbb{R}} x f(x) f'(x) \, dx. \tag{1.12}
\]

Exercise 1.23. Let \( f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \)

Show \( f \in C^\infty([0, 1]) \). Hints: you might start by first showing \( \lim_{t \to 0} f^{(n)}(t) = 0 \) for all \( n \in \mathbb{N} \).

Exercise 1.24. Show \( C^\infty_c(\mathbb{R}^d) \) is dense in \( L^p(\mathbb{R}^d, m) \) for any \( 1 \leq p < \infty \).

Exercise 1.25. If \( f \in L^1_{loc}(\mathbb{R}^d, m) \) and \( \varphi \in C^1(\mathbb{R}^d) \), then \( f \ast \varphi \in C^1(\mathbb{R}^d) \) and \( \partial_k(f \ast \varphi) = f \ast \partial_k \varphi \). Moreover if \( \varphi \in C^\infty_c(\mathbb{R}^d) \) then \( f \ast \varphi \in C^\infty(\mathbb{R}^d) \).

Exercise 1.26 (Integration by Parts). Suppose that \( (x, y) \in \mathbb{R} \times \mathbb{R}^d-1 \to f(x, y) \in \mathbb{C} \) and \( (x, y) \in \mathbb{R} \times \mathbb{R}^d-1 \to g(x, y) \in \mathbb{C} \) are measurable functions such that for each fixed \( y \in \mathbb{R}^d \), \( x \to f(x, y) \) and \( x \to g(x, y) \) are continuously differentiable. Also assume \( f \cdot g, \partial_x f \cdot g \) and \( f \cdot \partial_x g \) are integrable relative to Lebesgue measure on \( \mathbb{R} \times \mathbb{R}^d-1 \), where \( \partial_x f(x, y) := \frac{d}{dx}[f(x + t, y)]_{t=0} \). Show

\[
\int_{\mathbb{R} \times \mathbb{R}^d-1} \partial_x f(x, y) \cdot g(x, y) \, dx \, dy = -\int_{\mathbb{R} \times \mathbb{R}^d-1} f(x, y) \cdot \partial_x g(x, y) \, dx \, dy. \tag{1.13}
\]

(Note: this result and Fubini’s theorem proves Lemma ??.)

Hints: Let \( \psi \in C^\infty_c(\mathbb{R}) \) be a function which is 1 in a neighborhood of \( 0 \in \mathbb{R} \) and set \( \psi_\varepsilon(x) = \psi(\varepsilon x) \). First verify Eq. (1.13) with \( f(x, y) \) replaced by \( \psi_\varepsilon(x) f(x, y) \) by doing the \( x \) integral first. Then use the dominated convergence theorem to prove Eq. (1.13) by passing to the limit, \( \varepsilon \downarrow 0 \).

Exercise 1.27 (Wirtinger’s inequality, Folland 8.18). Given \( a > 0 \) and \( f \in C^1([0, a] \cap \mathbb{C}) \) such that \( f(0) = f(a) = 0 \), show

\[
\int_{0}^{a} |f(x)|^2 \, dx \leq \left( \frac{a}{\pi} \right)^2 \int_{0}^{a} |f'(x)|^2 \, dx.
\]

Hint: to use the notation above, let \( \pi L = a \) and extend \( f \) to \([-a, 0] \) by setting \( f(-x) = -f(x) \) for \( 0 \leq x \leq a \). Now compute \( \int_{0}^{a} |f(x)|^2 \, dx \) and \( \int_{0}^{a} |f'(x)|^2 \, dx \) in terms of their Fourier coefficients, \( \langle f \varphi_k \rangle_L \) and \( \langle f \varphi_k \rangle_L \) respectively.

Exercise 1.28. Let \( L = \sum_{|\alpha| \leq k} a_\alpha (x) \partial^\alpha \)

\[
(1.14)
\]

with \( a_\alpha \in \mathbb{P} \). Show \( L(S) \subset S \) and in particular \( \partial^\alpha f \) and \( x^\alpha f \) are back in \( S \) for all multi-indices \( \alpha \).

\[ ^2 \text{This means that } f \text{ and } g \text{ restricted to any bounded interval in } \mathbb{R} \text{ are absolutely continuous on that interval.} \]

\[ ^3 \text{You may assume the existence of such a } \psi, \text{ we will deal with this later.} \]

\[ ^4 \text{This means that } f \text{ restricted to any bounded interval in } \mathbb{R} \text{ are absolutely continuous on that interval.} \]
Exercise 1.29. In this problem let $d = 1$ so that $x, \xi \in \mathbb{R} = \mathbb{R}^1$. For any $m > 0$, show
\[
\mathcal{F}\left[e^{-m|x|}\right](\xi) = \frac{2m}{\sqrt{2\pi mn^2 + \xi^2}}
\]
and
\[
\mathcal{F}\left(\frac{1}{m^2 + \xi^2}\right)(x) = \frac{\sqrt{2\pi}}{2m} e^{-m|x|}.
\]
More precisely these equations mean;
\[
\mathcal{F}\left[x \to e^{-m|x|}\right](\xi) = \frac{2m}{\sqrt{2\pi mn^2 + \xi^2}}
\text{ and }
\mathcal{F}\left(\xi \to \frac{1}{m^2 + \xi^2}\right)(x) = \frac{\sqrt{2\pi}}{2m} e^{-m|x|}
\]
or equivalently,
\[
\mathcal{F}\left[e^{-m|\cdot|}\right] = \frac{2m}{\sqrt{2\pi mn^2 + (\cdot)^2}} \text{ and } \mathcal{F}\left(\frac{1}{m^2 + (\cdot)^2}\right) = \frac{\sqrt{2\pi}}{2m} e^{-m|\cdot|}.
\]

Exercise 1.30. Using the identity
\[
\frac{1}{\xi^2 + 1} = \int_0^\infty e^{-s(\xi^2 + 1)} ds
\]
along with Exercise 1.29 and the known Fourier transform of Gaussians to show
\[
e^{-|x|} = \int_0^\infty ds \frac{1}{\sqrt{\pi} s} e^{-s} e^{-\frac{x^2}{s}} \text{ for all } x \in \mathbb{R}. \tag{1.15}
\]
Thus we have written $e^{-|x|}$ as an average of Gaussians.

Exercise 1.31. Now let $x \in \mathbb{R}^d$ and $|x|^2 := \sum_{i=1}^d x_i^2$ be the standard Euclidean norm. Show for all $m > 0$ that
\[
\mathcal{F}\left[e^{-m|x|}\right](\xi) = \frac{2^{d/2}}{\sqrt{\pi}} \frac{m}{m^2 + |\xi|^2} \Gamma\left(\frac{d+1}{2}\right), \tag{1.16}
\]
where $\Gamma(x)$ in the gamma function defined by
\[
\Gamma(x) := \int_0^\infty t^x e^{-t} dt.
\]

Hint: By Exercise 1.30 with $x$ replaced by $m|x|$ we know that
\[
e^{-m|x|} = \int_0^\infty ds \frac{1}{\sqrt{\pi} s} e^{-s} e^{-\frac{m^2}{s}|x|^2} \text{ for all } x \in \mathbb{R}^d.
\]

Exercise 1.32. Suppose $p(\xi) = \sum_{|\alpha| \leq N} a_\alpha \xi^\alpha$ is a polynomial in $\xi \in \mathbb{R}^d$, and
\[
u \in L^2 \text{ such that } (p(\partial) u = g \in L^2 \text{ in the weak sense, i.e.}
\]
\[
\langle u, p(-\partial) \phi \rangle = \langle g, \phi \rangle \text{ for all } \phi \in C_c^\infty(\mathbb{R}^d). \tag{1.17}
\]
Show that Eq. (1.17) also holds for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. Hints: Let $\psi \in C_c^\infty(\mathbb{R}^d, [0,1])$ be chosen so that $\psi(x) = 1$ for $|x| \leq 1$ and for $n \in \mathbb{N}$, let $\psi_n(x) := \psi(x/n)$. Then for $\phi \in \mathcal{S}$, consider $\psi_n \cdot \phi$.

Exercise 1.33 (F.T. and Weak derivatives). Suppose $p(\xi) = \sum_{|\alpha| \leq N} a_\alpha \xi^\alpha$ is a polynomial in $\xi \in \mathbb{R}^d$ and $f, g \in L^2(m)$. Show $p(\partial) f = g$ weakly iff $p(ik) \hat{f}(k) = \hat{g}(k)$ for a.e. $k$.

Exercise 1.34. Show for $f \in \mathcal{S}(\mathbb{R})$ that;

1. For all $x \in \mathbb{R}$,
\[
|f(x)| \leq \frac{1}{\sqrt{2\pi}} \left\| \hat{f} \right\|_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{f}(k)| \, dk
\]
and
\[
|f(x)| \leq \frac{1}{\sqrt{2}} \left[ \int_{\mathbb{R}} |\hat{f}(k)|^2 \left(1 + k^2\right) \, dk\right]^{1/2}.
\]
2. Use the last displayed inequality and the basic properties of the Fourier transform to prove the “Sobolev inequality,”
\[
|f(x)|^2 \leq \frac{1}{2} \left[ \|f\|_2^2 + \|f’\|_2^2 \right] \text{ for all } x \in \mathbb{R},
\]
where
\[
\|f\|_2^2 := \int_{\mathbb{R}} |f(x)|^2 \, dx.
\]

Exercise 1.35 (Sampling Theorem). Let
\[
\text{sinc } x = \begin{cases} \sin \frac{\pi x}{\pi x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}
\]
and for any $a \in (0, \infty)$, let
\[
\mathcal{H}_a = \{f \in L^2(m) : \hat{f}(\xi) = 0 \text{ a.e. when } |\xi| > \pi a\}.
\]
Show
1. Show that every \( f \in H_a \) has a version\(^6\) \( f_0 \in C_0 (\mathbb{R}) \) and moreover,

\[
\| f_0 \|_u \leq \sqrt{a} \| f \|_{L^2(m)}.
\]

(1.18)

[We now identify \( f \) with this continuous version.] **Hint:** after identifying \( L^2 ([-\pi a, \pi a], \lambda) \) as a subspace of \( L^2 (\mathbb{R}, \lambda) \) one has

\[
H_a = \mathcal{F}^{-1} L^2 ([\pi a, \pi a], \lambda).
\]

2. Show by direct computation that

\[
\mathcal{F}^{-1} \left[ \frac{1}{\sqrt{2\pi a}} e^{-i\xi/a} 1_{|\xi| \leq \pi a} \right] (x) = \text{sinc} (ax - n).
\]

3. If \( f \in H_a \) then (assuming \( f \) is the \( C_0 \) – version as in part a), show

\[
f(x) = \sum_{k=-\infty}^{\infty} f(k/a) \text{sinc}(ax - k),
\]

where the series converges both uniformly and in \( L^2 \). **Hint:** Start by writing \( \hat{f}(\xi) \) for \( |\xi| \leq \pi a \) as a Fourier expansion in the orthonormal basis \( \{e^{-i\xi/a}\}_{n=-\infty}^{\infty} \) for \( L^2 ([-\pi a, \pi a], \frac{m}{\pi a}) \).

In the terminology of signal analysis, a signal of band width \( 2\pi a \) is completely determined by sampling its value at a sequence of points \( \{k/2\pi a\} \) whose spacing is the reciprocal of the bandwidth.

**Exercise 1.36.** Let \( \lambda := (2\pi)^{-d/2} m \) where \( m \) is Lebesgue measure on \( \mathbb{R}^d \).

Suppose that \( f \in L^2 (\lambda) \) such that \( f = f 1_S \) a.e. for some \( S \in \mathcal{B}_{\mathbb{R}^d} \) with \( \lambda(S) < \infty \). Show for every \( E \in \mathcal{B}_{\mathbb{R}^d} \) that

\[
\int_E |\hat{f}|^2 \ d\lambda \leq \| f \|^2_{L^2(\lambda)} \lambda(S) \cdot \lambda(E).
\]

(The Fourier transform of a function whose support has finite measure.)

---

\(^6\) We say that \( f_0 \) is a version of \( f \) if \( f(x) = f_0(x) \) for \( m - \text{a.e.} \ x \).