# 0

## Math 240C Homework Problem List for S2018

### 0.1 Homework C1. Due Friday, April 6, 2018

- **Hand in**: 1.3, 1.4, 1.5, 1.25, 1.26
- **Look at**: 1.23

### 0.2 Homework C2. Due Friday, April 13, 2018 ($L^p$ inequalities)

- **Hand in**: 1.1, 1.2, 1.6, 1.7, 1.19, 1.20
- **Look at**: 1.21, 1.22

### 0.3 Homework C3. Due Friday, April 20, 2018 (Fourier Series problems)

- **Hand in**: 1.8, 1.10, 1.11, 1.12, 1.13, 1.14, 1.16
- **Look at**: 1.24 (done in class), 1.9, 1.15

### 0.4 Homework C4. Due Friday, April 27, 2018 (Fourier Transform problems)

- **Hand in**: 1.27, 1.29, 1.30, 1.31, 1.34, 1.35
- **Look at**: 1.28

### 0.5 Homework C5. Due Friday, May 4, 2018

- **Hand in**: 1.32, 1.33, 1.35, 1.36

# Problems to Solve

**Exercise 1.1.** If $(X, \rho)$ is a metric space and $\mu$ is a **finite** measure on $(X, \mathcal{B}_X)$, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $\mu(V \setminus F) = \mu(F \triangle V) < \varepsilon$.

You may find information in the supplement helpful for this problem. Here are some more suggestions.

1. Let $\mathcal{B}_0$ denote those $A \subset X$ such that for all $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $d_\mu(F, V) = \mu(V \setminus F) < \varepsilon$.
2. Show $\mathcal{B}_0$ contains all closed (or open if you like).
3. Show $\mathcal{B}_0$ is a $\sigma$-algebra.
4. Explain why this proves the result.

**Exercise 1.2.** Let $(X, \rho)$ be a metric space and $\mu$ be a measure on $(X, \mathcal{B}_X)$. If there exists open sets, $\{V_n\}_{n=1}^\infty$, of $X$ such that $V_n \uparrow X$ and $\mu(V_n) < \infty$ for all $n$, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $d_\mu(F, V) = \mu(V \setminus F) < \varepsilon$. **Hints:**

1. Show it suffices to prove; for all $\varepsilon > 0$ and $A \in \mathcal{B}_X$, there exists an open set $V \subset X$ such that $A \subset V$ and $\mu(V \setminus A) < \varepsilon$.
2. Now you must verify the assertion above holds. For this, you may find it useful to apply Exercise 1.1 to the measures, $\mu_n : \mathcal{B}_X \to [0, \mu(V_n)]$, defined by $\mu_n(A) := \mu(A \cap V_n)$ for all $A \in \mathcal{B}_X$. The $\varepsilon$ in Exercise 1.1 should be replaced by judiciously chosen small quantities depending on $n$.

**Exercise 1.3 (Folland Problem 2.62 on p. 80. ).** Rotation invariance of surface measure on $S^{n-1}$.

**Exercise 1.4 (Folland Problem 2.64 on p. 80. ).** On the integrability of $|x|^a |\log |x||^b$ for $x$ near 0 and $x$ near $\infty$ in $\mathbb{R}^n$.

**Exercise 1.5.** Show, using Problem 1.3 that

$$\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega) = \frac{1}{d} \delta_{ij} \sigma\left(S^{d-1}\right).$$

**Hint:** show $\int_{S^{d-1}} \omega_i^2 d\sigma(\omega)$ is independent of $i$ and therefore

$$\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d} \sum_{j=1}^d \int_{S^{d-1}} \omega_j^2 d\sigma(\omega).$$

**Exercise 1.6 (Folland 6.38 on p. 199.).** Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $f : X \to \mathbb{C}$ is a measurable function, $0 < p < \infty$, $\lambda_f(\alpha) := \mu(|f| > \alpha)$ for all $\alpha \in (0, \infty)$, and

$$M_p(f) := \sum_{k=-\infty}^\infty 2^{kp} \lambda_f\left(2^k\right).$$

Show

$$\left(1 - 2^{-p}\right) M_p(f) \le \int_X |f|^p \, d\mu \le 2^p M_p(f) \tag{1.1}$$

which then implies $f \in L^p(\mu)$ iff $M_p(f) < \infty$.

**Hint:** first note that

$$\int_X |f|^p \, d\mu = \sum_{k \in \mathbb{Z}} \int_{\{2^k < |f| \le 2^{k+1}\}} |f|^p \, d\mu. \tag{1.2}$$

**Exercise 1.7 (Folland 6.39 on p. 199.).** Suppose $(X, \mathcal{M}, \mu)$ is a measure space, $f : X \to \mathbb{C}$ is a measurable function, $0 < p < \infty$, and $f \in L^p(\mu)$. Show

$$\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0 = \lim_{\alpha \to \infty} \alpha^p \lambda_f(\alpha).$$

**Hint:** for the limit, $\lim_{\alpha \to 0} \alpha^p \lambda_f(\alpha) = 0$, start with the special case where $f$ is a simple function.

**Exercise 1.8.** Show $\sum_{k=1}^\infty k^{-2} = \pi^2/6$, by taking $f(x) = x$ on $[-\pi, \pi]$ and computing $\|f\|_2^2$ directly and then in terms of the Fourier Coefficients $\tilde{f}$ of $f$.

**Exercise 1.9 (Riemann Lebesgue Lemma for Fourier Series).** Show for $f \in L^1\left([-\pi, \pi]^d\right)$ that $\tilde{f} \in c_0(\mathbb{Z}^d)$, i.e. $\tilde{f} : \mathbb{Z}^d \to \mathbb{C}$ and $\lim_{k \to \infty} \tilde{f}(k) = 0$. **Hint:** If $f \in L^2\left([-\pi, \pi]^d\right)$, this follows from Bessel's inequality. Now use a density argument.

**Exercise 1.10.** Suppose $f \in L^1([-\pi, \pi]^d)$ is a function such that $\tilde{f} \in \ell^1(\mathbb{Z}^d)$ and set

$$g(x) := \sum_{k \in \mathbb{Z}^d} \tilde{f}(k) e^{ik \cdot x} \text{ (pointwise)}.$$

1. Show $g \in C_{per}\left(\mathbb{R}^d\right)$.
2. Show $g(x) = f(x)$ for $m$ – a.e. $x$ in $[-\pi,\pi]^d$. **Hint**: Show $\tilde{g}(k) = \tilde{f}(k)$ and apply Exercise **??** or results proved in class.
3. Conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$ and in particular $f \in L^p([-\pi,\pi]^d)$ for all $p \in [1,\infty]$.

**Exercise 1.11 (Smoothness implies decay).** We use the following notation below.

**Notation:** Given a multi-index $\alpha \in \mathbb{Z}_+^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$,

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \text{ and } \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$

Further for $k \in \mathbb{N}_0$, let $f \in C_{per}^k\left(\mathbb{R}^d\right)$ iff $f \in C^k\left(\mathbb{R}^d\right) \cap C_{per}\left(\mathbb{R}^d\right)$, $\partial_x^\alpha f(x)$ exists and is continuous for $|\alpha| \le k$.

Suppose $m \in \mathbb{N}_0$, $\alpha$ is a multi-index such that $|\alpha| \le 2m$ and $f \in C_{per}^{2m}\left(\mathbb{R}^d\right)$[1].

1. Using integration by parts, show (using Notation **??**) that
$$(ik)^\alpha \tilde{f}(k) = \langle \partial^\alpha f | \varphi_k \rangle \text{ for all } k \in \mathbb{Z}^d.$$
Note: This equality implies
$$\left|\tilde{f}(k)\right| \le \frac{1}{k^\alpha}\left\|\partial^\alpha f\right\|_H \le \frac{1}{k^\alpha}\left\|\partial^\alpha f\right\|_\infty.$$
2. Now let $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_i^2$, Working as in part 1) show
$$\langle (1-\Delta)^m f | \varphi_k \rangle = (1 + \|k\|^2)^m \tilde{f}(k). \tag{1.3}$$
where $\|k\|^2 = \sum_{j=1}^d k_j^2$.

**Exercise 1.12 (A Sobolev Imbedding Theorem).** Suppose $s \in \mathbb{R}$ and $\left\{c_k \in \mathbb{C} : k \in \mathbb{Z}^d\right\}$ are coefficients such that

$$\sum_{k \in \mathbb{Z}^d} |c_k|^2 (1 + |k|^2)^s < \infty.$$

Show if $s > \frac{d}{2} + m$, the function $f$ defined by

$$f(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ik\cdot x}$$

is in $C_{per}^m\left(\mathbb{R}^d\right)$. **Hint**: Work as in the above remark to show

$$\sum_{k \in \mathbb{Z}^d} |c_k|\,|k^\alpha| < \infty \text{ for all } |\alpha| \le m.$$

**Exercise 1.13 (Poisson Summation Formula).** Let $F \in L^1\left(\mathbb{R}^d\right)$,

$$E := \left\{x \in \mathbb{R}^d : \sum_{k \in \mathbb{Z}^d} |F(x + 2\pi k)| = \infty\right\}$$

and set (see Section **??** below for an introduction to the Fourier transform)

$$\hat{F}(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} F(x)\, e^{-ik\cdot x} dx \text{ for } k \in \mathbb{Z}^d.$$

Further **assume** $\hat{F} \in \ell^1(\mathbb{Z}^d)$. [This can be achieved by assuming $F$ is sufficiently differentiable with the derivatives being integrable like in Exercise 1.11.]

1. Show $m(E) = 0$ and $E + 2\pi k = E$ for all $k \in \mathbb{Z}^d$. **Hint**: Compute $\int_{[-\pi,\pi]^d} \sum_{k \in \mathbb{Z}^d} |F(x + 2\pi k)|\, dx$.
2. Let
$$f(x) := \begin{cases} \sum_{k \in \mathbb{Z}^d} F(x + 2\pi k) & \text{for } x \notin E \\ 0 & \text{if } x \in E. \end{cases}$$
Show $f \in L^1([-\pi,\pi]^d)$ and $\tilde{f}(k) = (2\pi)^{-d/2}\hat{F}(k)$.
3. Using item 2) and the assumptions on $F$, show
$$f(x) = \sum_{k \in \mathbb{Z}^d} \tilde{f}(k) e^{ik\cdot x} = \sum_{k \in \mathbb{Z}^d} (2\pi)^{-d/2}\hat{F}(k) e^{ik\cdot x} \text{ for } m - \text{a.e. } x,$$
i.e.
$$\sum_{k \in \mathbb{Z}^d} F(x + 2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{F}(k) e^{ik\cdot x} \text{ for } m - \text{a.e. } x \tag{1.4}$$
and form this conclude that $f \in L^1([-\pi,\pi]^d) \cap L^\infty([-\pi,\pi]^d)$.
**Hint:** see the hint for item 2. of Exercise 1.10.
4. Suppose we now assume that $F \in C\left(\mathbb{R}^d\right)$ and $F$ satisfies $|F(x)| \le C(1 + |x|)^{-s}$ for some $s > d$ and $C < \infty$. Under these added assumptions on $F$, show Eq. (1.4) holds for **all** $x \in \mathbb{R}^d$ and in particular
$$\sum_{k \in \mathbb{Z}^d} F(2\pi k) = (2\pi)^{-d/2} \sum_{k \in \mathbb{Z}^d} \hat{F}(k).$$

[1] We view $C_{per}(\mathbb{R}^d)$ as a subspace of $H = L^2\left([-\pi,\pi]^d\right)$ by identifying $f \in C_{per}(\mathbb{R}^d)$ with $f|_{[-\pi,\pi]^d} \in H$.

**Exercise 1.14 (Heat Equation 1.).** Let $[0,\infty)\times\mathbb{R}\ni(t,x)\to u(t,x)$ be a continuous function such that $u(t,\cdot)\in C_{per}(\mathbb{R})$ for all $t\geq 0$, $\dot{u}:=u_t$, $u_x$, and $u_{xx}$ exists and are continuous when $t>0$. Further assume that $u$ satisfies the heat equation $\dot{u}=\frac{1}{2}u_{xx}$. Let $\tilde{u}(t,k):=\langle u(t,\cdot)|\varphi_k\rangle$ for $k\in\mathbb{Z}$. Show for $t>0$ and $k\in\mathbb{Z}$ that $\tilde{u}(t,k)$ is differentiable in $t$ and $\frac{d}{dt}\tilde{u}(t,k)=-k^2\tilde{u}(t,k)/2$. Use this result to show

$$u(t,x)=\sum_{k\in\mathbb{Z}}e^{-\frac{t}{2}k^2}\tilde{f}(k)e^{ikx} \tag{1.5}$$

where $f(x):=u(0,x)$ and as above

$$\tilde{f}(k)=\langle f|\varphi_k\rangle=\int_{-\pi}^{\pi}f(y)\,e^{-iky}dy=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,e^{-iky}dm(y).$$

Notice from Eq. (1.5) that $(t,x)\to u(t,x)$ is $C^\infty$ for $t>0$.

**Exercise 1.15 (Heat Equation 2.).** Let $q_t(x):=\frac{1}{2\pi}\sum_{k\in\mathbb{Z}}e^{-\frac{t}{2}k^2}e^{ikx}$. Show;

1. Eq. (1.5) may be rewritten as
$$u(t,x)=\int_{-\pi}^{\pi}q_t(x-y)f(y)\,dy$$
and
2. $q_t(x)$ may be expresses
$$q_t(x)=\sum_{k\in\mathbb{Z}}p_t(x+k2\pi)$$
where $p_t(x):=\frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}x^2}$.
3. Also show $u(t,x)$ may be written as
$$u(t,x)=(p_t*f)(x):=\int_{\mathbb{R}^d}p_t(x-y)f(y)\,dy.$$

**Hint**: To show $q_t(x)=\sum_{k\in\mathbb{Z}}p_t(x+k2\pi)$, use the Poisson summation formula (Exercise 1.13) and the Gaussian integration identity,

$$\hat{p}_t(\omega)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}p_t(x)\,e^{i\omega x}dx=\frac{1}{\sqrt{2\pi}}e^{-\frac{t}{2}\omega^2}. \tag{1.6}$$

**Exercise 1.16 (Wave Equation).** Let $u\in C^2(\mathbb{R}\times\mathbb{R})$ be such that $u(t,\cdot)\in C_{per}(\mathbb{R})$ for all $t\in\mathbb{R}$. Further assume that $u$ solves the wave equation, $u_{tt}=u_{xx}$. Let $f(x):=u(0,x)$ and $g(x)=\dot{u}(0,x)$. Show $\tilde{u}(t,k):=\langle u(t,\cdot),\varphi_k\rangle$ for $k\in\mathbb{Z}$ is twice continuously differentiable in $t$ and $\frac{d^2}{dt^2}\tilde{u}(t,k)=-k^2\tilde{u}(t,k)$. Use this result to show

$$u(t,x)=\sum_{k\in\mathbb{Z}}\left(\tilde{f}(k)\cos(kt)+\tilde{g}(k)\frac{\sin kt}{k}\right)e^{ikx} \tag{1.7}$$

with the sum converging absolutely. Also show that $u(t,x)$ may be written as

$$u(t,x)=\frac{1}{2}\left[f(x+t)+f(x-t)\right]+\frac{1}{2}\int_{-t}^{t}g(x+\tau)d\tau. \tag{1.8}$$

**Hint**: To show Eq. (1.7) implies (1.8) use

$$\cos kt=\frac{e^{ikt}+e^{-ikt}}{2},$$
$$\sin kt=\frac{e^{ikt}-e^{-ikt}}{2i},\text{ and}$$
$$\frac{e^{ik(x+t)}-e^{ik(x-t)}}{ik}=\int_{-t}^{t}e^{ik(x+\tau)}d\tau.$$

**Exercise 1.17.** Let $f\in L^1((-\pi,\pi])$ which we extend to a $2\pi$ – periodic function on $\mathbb{R}$ and continue to denote by $f$. If there exists $q\in\mathbb{N}$ such that $f\left(x+\frac{2\pi}{q}\right)=f(x)$ for $m$ – a.e. $x$, then $\tilde{f}(k)=0$ unless $q$ divides $k$.

**Exercise 1.18.** In this problem we assume the notation from subsection **??** with $d=1$. For simplicity of notation we identify $L^2((-\pi,\pi],d\theta)$ with $2\pi$ – periodic functions on $\mathbb{R}$ via,

$$L^2((-\pi,\pi],d\theta)\ni f\longleftrightarrow\sum_{n\in\mathbb{Z}}f(x+n2\pi)\,1_{(-\pi,\pi]}(x+n2\pi)\in L^2_{per}(\mathbb{R}).$$

Given $\alpha\in\mathbb{R}$ let $(U_\alpha f)(\theta)=f(\theta+\alpha2\pi)$ wherein we have used the above identification. If $\alpha\notin\mathbb{Q}$ show

$$M_\alpha=\mathrm{Nul}(U_\alpha-I)=\mathbb{C}\cdot 1.$$

If $\alpha\in\mathbb{Q}$ write $\alpha=\frac{p}{q}$ where $\gcd(q,p)=1$, i.e. $p$ and $q$ are relatively prime. In this case show $M_\alpha=\mathrm{Nul}(U_\alpha-I)$ consists of those $f\in L^2_{per}(\mathbb{R})$ such that $f\left(x+\frac{2\pi}{q}\right)=f(x)$ for $m$ – a.e. $x$. [Consequently, combining this exercise with Mean Ergodic Theorem **??** shows,

$$\frac{1}{n}\sum_{k=0}^{n-1}U_\alpha^k\xrightarrow{s}P_{M_\alpha}$$

where $M_\alpha$ depends on $\alpha$ as described above.]

**Exercise 1.19 (Folland 6.27 on p. 196. Hilbert's Inequality). Hint**: See Theorem **??** which is Theorem 6.20 in Folland .

**Exercise 1.20 (Folland 6.22).** Exercise, Folland 6.22 on p. 192.

**Exercise 1.21 (Global Integration by Parts Formula).** Suppose that $f, g : \mathbb{R} \to \mathbb{C}$ are locally absolutely continuous functions[2] such that $f'g$, $fg'$, and $fg$ are all Lebesgue integrable functions on $\mathbb{R}$. Prove the following integration by parts formula;

$$\int_{\mathbb{R}} f'(x) \cdot g(x)\, dx = -\int_{\mathbb{R}} f(x) \cdot g'(x)\, dx. \tag{1.9}$$

Similarly show that; if $f, g : [0,\infty) \to [0,\infty)$ are locally absolutely continuous functions such that $f'g$, $fg'$, and $fg$ are all Lebesgue integrable functions on $[0,\infty)$, then

$$\int_0^\infty f'(x) \cdot g(x)\, dx = -f(0)\, g(0) - \int_0^\infty f(x) \cdot g'(x)\, dx. \tag{1.10}$$

**Outline:** 1. First use the theory developed to see that Eq. (1.9) holds if $f(x) = 0$ for $|x| \geq N$ for some $N < \infty$.

2. Let $\psi : \mathbb{R} \to [0,1]$ be a continuously differentiable function such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$.[3] For any $\varepsilon > 0$ let $\psi_\varepsilon(x) = \psi(\varepsilon x)$ Write out the identity in Eq. (1.9) with $f(x)$ being replaced by $f(x)\psi_\varepsilon(x)$.

3. Now use the dominated convergence theorem to pass to the limit as $\varepsilon \downarrow 0$ in the identity you found in step 2.

4. A similar outline works to prove Eq. (1.10).

**Exercise 1.22 (Heisenberg's Inequality).** Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a locally absolutely continuous function[4], show

$$\int_{\mathbb{R}} |f(x)|^2\, dx \leq 2 \left[\int_{\mathbb{R}} |xf(x)|^2\, dx\right]^{1/2} \left[\int_{\mathbb{R}} |f'(x)|^2\, dx\right]^{1/2}. \tag{1.11}$$

**Hint:** assuming the right hand side of the above inequality is finite show

$$\int_{\mathbb{R}} |f(x)|^2\, dx = -2\,\mathrm{Re} \int_{\mathbb{R}} x\bar{f}(x)\, f'(x)\, dx. \tag{1.12}$$

[2] This means that $f$ and $g$ restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.

[3] You may assume the existence of such a $\psi$, we will deal with this later.

[4] This means that $f$ restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.

**Exercise 1.23.** Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0,1])$. **Hints:** you might start by first showing $\lim_{t\downarrow 0} f^{(n)}(t) = 0$ for all $n \in \mathbb{N}_0$.

**Exercise 1.24.** Show $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d, m)$ for any $1 \leq p < \infty$.

**Exercise 1.25.** If $f \in L^1_{loc}(\mathbb{R}^d, m)$ and $\varphi \in C_c^1(\mathbb{R}^d)$, then $f * \varphi \in C^1(\mathbb{R}^d)$ and $\partial_i(f*\varphi) = f * \partial_i \varphi$. Moreover if $\varphi \in C_c^\infty(\mathbb{R}^d)$ then $f * \varphi \in C^\infty(\mathbb{R}^d)$.

**Exercise 1.26 (Integration by Parts).** Suppose that $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to f(x,y) \in \mathbb{C}$ and $(x,y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to g(x,y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^d$, $x \to f(x,y)$ and $x \to g(x,y)$ are continuously differentiable. Also assume $f \cdot g$, $\partial_x f \cdot g$ and $f \cdot \partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R} \times \mathbb{R}^{d-1}$, where $\partial_x f(x,y) := \frac{d}{dt} f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} \partial_x f(x,y) \cdot g(x,y) dx dy = -\int_{\mathbb{R}\times\mathbb{R}^{d-1}} f(x,y) \cdot \partial_x g(x,y) dx dy. \tag{1.13}$$

(Note: this result and Fubini's theorem proves Lemma **??**.)

**Hints:** Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0 \in \mathbb{R}$ and set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. First verify Eq. (1.13) with $f(x,y)$ replaced by $\psi_\varepsilon(x) f(x,y)$ by doing the $x-$ integral first. Then use the dominated convergence theorem to prove Eq. (1.13) by passing to the limit, $\varepsilon \downarrow 0$.

**Exercise 1.27 (Wirtinger's inequality, Folland 8.18).** Given $a > 0$ and $f \in C^1([0,a], \mathbb{C})$ such that $f(0) = f(a) = 0$, show[5]

$$\int_0^a |f(x)|^2\, dx \leq \left(\frac{a}{\pi}\right)^2 \int_0^a |f'(x)|^2\, dx.$$

**Hint:** to use the notation above, let $\pi L = a$ and extend $f$ to $[-a, 0]$ by setting $f(-x) = -f(x)$ for $0 \leq x \leq a$. Now compute $\int_0^a |f(x)|^2\, dx$ and $\int_0^a |f'(x)|^2\, dx$ in terms of their Fourier coefficients, $\langle f|\varphi_k^L\rangle_L$ and $\langle f'|\varphi_k^L\rangle_L$ respectively.

**Exercise 1.28.** Let

$$L = \sum_{|\alpha|\leq k} a_\alpha(x)\, \partial^\alpha \tag{1.14}$$

with $a_\alpha \in \mathcal{P}$. Show $L(\mathcal{S}) \subset \mathcal{S}$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $\mathcal{S}$ for all multi-indices $\alpha$.

[5] This inequality is sharp as is seen by taking $f(x) = \sin(\pi x/a)$.

**Exercise 1.29.** In this problem let $d=1$ so that $x,\xi\in\mathbb{R}=\mathbb{R}^1$. For any $m>0$, show

$$\mathcal{F}\left[e^{-m|x|}\right](\xi)=\frac{2m}{\sqrt{2\pi}}\frac{1}{m^2+\xi^2}$$

and

$$\mathcal{F}\left(\frac{1}{m^2+\xi^2}\right)(x)=\frac{\sqrt{2\pi}}{2m}e^{-m|x|}.$$

More precisely these equations mean;

$$\mathcal{F}\left[x\to e^{-m|x|}\right](\xi)=\frac{2m}{\sqrt{2\pi}}\frac{1}{m^2+\xi^2}\text{ and}$$
$$\mathcal{F}\left(\xi\to\frac{1}{m^2+\xi^2}\right)(x)=\frac{\sqrt{2\pi}}{2m}e^{-m|x|}$$

or equivalently,

$$\mathcal{F}\left[e^{-m|\cdot|}\right]=\frac{2m}{\sqrt{2\pi}}\frac{1}{m^2+(\cdot)^2}\text{ and }\mathcal{F}\left(\frac{1}{m^2+(\cdot)^2}\right)=\frac{\sqrt{2\pi}}{2m}e^{-m|\cdot|}.$$

**Exercise 1.30.** Using the identity

$$\frac{1}{\xi^2+1}=\int_0^\infty e^{-s(\xi^2+1)}ds$$

along with Exercise 1.29 and the known Fourier transform of Gaussians to show

$$e^{-|x|}=\int_0^\infty ds\frac{1}{\sqrt{\pi s}}e^{-s}e^{-\frac{x^2}{4s}}\text{ for all }x\in\mathbb{R}.\tag{1.15}$$

Thus we have written $e^{-|x|}$ as an average of Gaussians.

**Exercise 1.31.** Now let $x\in\mathbb{R}^d$ and $|x|^2:=\sum_{i=1}^d x_i^2$ be the standard Euclidean norm. Show for all $m>0$ that

$$\mathcal{F}\left[e^{-m|x|}\right](\xi)=\frac{2^{d/2}}{\sqrt{\pi}}\Gamma\left(\frac{d+1}{2}\right)\frac{m}{\left(m^2+|\xi|^2\right)^{\frac{d+1}{2}}},\tag{1.16}$$

where $\Gamma(x)$ in the gamma function defined as

$$\Gamma(x):=\int_0^\infty t^x e^{-t}\frac{dt}{t}.$$

**Hint:** By Exercise 1.30 with $x$ replaced by $m|x|$ we know that

$$e^{-m|x|}=\int_0^\infty ds\frac{1}{\sqrt{\pi s}}e^{-s}e^{-\frac{m^2}{4s}|x|^2}\text{ for all }x\in\mathbb{R}^d.$$

**Exercise 1.32.** Suppose $p(\xi)=\sum_{|\alpha|\le N}a_\alpha\xi^\alpha$ is a polynomial in $\xi\in\mathbb{R}^d$, and $u\in L^2$ such that $p(\partial)u=g\in L^2$ in the weak sense, i.e.

$$\langle u,p(-\partial)\varphi\rangle=\langle g,\varphi\rangle\text{ for all }\varphi\in C_c^\infty\left(\mathbb{R}^d\right).\tag{1.17}$$

Show that Eq. (1.17) also holds for all $\varphi\in\mathcal{S}\left(\mathbb{R}^d\right)$. **Hints:** Let $\psi\in C_c^\infty\left(\mathbb{R}^d,[0,1]\right)$ be chosen so that $\psi(x)=1$ for $|x|\le 1$ and for $n\in\mathbb{N}$, let $\psi_n(x):=\psi(x/n)$. Then for $\varphi\in\mathcal{S}$, consider $\psi_n\cdot\varphi$.

**Exercise 1.33 (F.T. and Weak derivatives).** Suppose $p(\xi)=\sum_{|\alpha|\le N}a_\alpha\xi^\alpha$ is a polynomial in $\xi\in\mathbb{R}^d$ and $f,g\in L^2(m)$. Show $p(\partial)f=g$ weakly iff $p(ik)\hat{f}(k)=\hat{g}(k)$ for a.e. $k$.

**Exercise 1.34.** Show for $f\in\mathcal{S}(\mathbb{R})$ that;

1. For all $x\in\mathbb{R}$,

$$|f(x)|\le\frac{1}{\sqrt{2\pi}}\left\|\hat{f}\right\|_1=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|\hat{f}(k)\right|dk$$

and

$$|f(x)|\le\frac{1}{\sqrt{2}}\left[\int_{\mathbb{R}}\left|\hat{f}(k)\right|^2\left(1+k^2\right)dk\right]^{1/2}.$$

2. Use the last displayed inequality and the basic properties of the Fourier transform to prove the "Sobolev inequality,"

$$|f(x)|^2\le\frac{1}{2}\left[\|f\|_2^2+\|f'\|_2^2\right]\text{ for all }x\in\mathbb{R},$$

where

$$\|f\|_2^2:=\int_{\mathbb{R}}|f(x)|^2dx.$$

**Exercise 1.35 (Sampling Theorem).** Let

$$\operatorname{sinc}x=\begin{cases}\frac{\sin\pi x}{\pi x} & \text{if }x\neq 0\\ 1 & \text{if }x=0\end{cases}$$

and for any $a\in(0,\infty)$, let

$$\mathcal{H}_a=\{f\in L^2(m):\hat{f}(\xi)=0\text{ a.e. when }|\xi|>\pi a\}.$$

Show

1. Show that every $f \in \mathcal{H}_a$ has a version[6] $f_0 \in C_0(\mathbb{R})$ and moreover,

$$\|f_0\|_u \leq \sqrt{a}\, \|f\|_{L^2(m)} \,. \tag{1.18}$$

[We now identify $f$ with this continuous version.] **Hint:** after identifying $L^2([-\pi a, \pi a], \lambda)$ as a subspace of $L^2(\mathbb{R}, \lambda)$ one has

$$\mathcal{H}_a = \mathcal{F}^{-1} L^2([-\pi a, \pi a], \lambda)\,.$$

2. Show by direct computation that

$$\mathcal{F}^{-1}\left[\frac{1}{\sqrt{2\pi a}} e^{-in\xi/a} 1_{|\xi| \leq \pi a}\right](x) = \operatorname{sinc}(ax - n)\,.$$

3. If $f \in \mathcal{H}_a$ then (assuming $f$ is the $C_0$ – version as in part a), show

$$f(x) = \sum_{k=-\infty}^{\infty} f(k/a) \operatorname{sinc}(ax - k),$$

where the series converges both uniformly and in $L^2$. [**Hint:** Start by writing $\hat{f}(\xi)$ for $|\xi| \leq \pi a$ as a Fourier expansion in the orthonormal basis $\left\{e^{-in\xi/a}\right\}_{n=-\infty}^{\infty}$ for $L^2\left([-\pi a, \pi a], \frac{m}{2\pi a}\right)$.]

In the terminology of signal analysis, a signal of band width $2\pi a$ is completely determined by sampling its value at a sequence of points $\{k/2\pi a\}$ whose spacing is the reciprocal of the bandwidth.

**Exercise 1.36.** Let $\lambda := (2\pi)^{-d/2} m$ where $m$ is Lebesgue measure on $\mathbb{R}^d$. Suppose that $f \in L^2(\lambda)$ such that $f = f 1_S$ a.e. for some $S \in \mathcal{B}_{\mathbb{R}^d}$ with $\lambda(S) < \infty$. Show for ever $E \in \mathcal{B}_{\mathbb{R}^d}$ that

$$\int_E \left|\hat{f}\right|^2 d\lambda \leq \|f\|_{L^2(\lambda)}^2 \lambda(S) \cdot \lambda(E)\,.$$

(The Fourier transform of a function whose support has finite measure.)

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[6] We say that $f_0$ is a version of $f$ if $f(x) = f_0(x)$ for $m$ – a.e. $x$.