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Math 240C Homework Problem List for S2018

0.1 Homework C1. Due Friday, April 6, 2018

- **Hand in**: 1.1, 1.2, 1.3, 1.5, 1.6
- **Look at**: 1.4

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Problems to Solve

Exercise 1.1 (Folland Problem 2.62 on p. 80.). Rotation invariance of surface measure on S^{n-1}.

Exercise 1.2 (Folland Problem 2.64 on p. 80.). On the integrability of $|x|^a \left|\log|x|\right|^b$ for x near 0 and x near ∞ in $\mathbb{R}^n$.

Exercise 1.3. Show, using Problem 1.1 that

$$\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega) = \frac{1}{d}\delta_{ij}\sigma\left(S^{d-1}\right).$$

Hint: show $\int_{S^{d-1}} \omega_i^2 d\sigma(\omega)$ is independent of i and therefore

$$\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d}\sum_{j=1}^{d}\int_{S^{d-1}} \omega_j^2 d\sigma(\omega).$$

Exercise 1.4. Let

$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0,1])$. **Hints:** you might start by first showing $\lim_{t\downarrow 0} f^{(n)}(t) = 0$ for all $n \in \mathbb{N}_0$.

Exercise 1.5. If $f \in L^1_{loc}(\mathbb{R}^d, m)$ and $\varphi \in C_c^1\left(\mathbb{R}^d\right)$, then $f * \varphi \in C^1\left(\mathbb{R}^d\right)$ and $\partial_i(f*\varphi) = f * \partial_i\varphi$. Moreover if $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$ then $f*\varphi \in C^\infty\left(\mathbb{R}^d\right)$.

Exercise 1.6 (Integration by Parts). Suppose that $(x,y) \in \mathbb{R}\times\mathbb{R}^{d-1} \to f(x,y) \in \mathbb{C}$ and $(x,y) \in \mathbb{R}\times\mathbb{R}^{d-1} \to g(x,y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^d$, $x \to f(x,y)$ and $x \to g(x,y)$ are continuously differentiable. Also assume $f\cdot g$, $\partial_x f \cdot g$ and $f\cdot\partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R}\times\mathbb{R}^{d-1}$, where $\partial_x f(x,y) := \frac{d}{dt}f(x+t,y)|_{t=0}$. Show

$$\int_{\mathbb{R}\times\mathbb{R}^{d-1}} \partial_x f(x,y)\cdot g(x,y)dxdy = -\int_{\mathbb{R}\times\mathbb{R}^{d-1}} f(x,y)\cdot\partial_x g(x,y)dxdy. \tag{1.1}$$

Hints: Let $\psi \in C_c^\infty(\mathbb{R})$ be a function which is 1 in a neighborhood of $0 \in \mathbb{R}$ and set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. First verify Eq. (1.1) with $f(x,y)$ replaced by $\psi_\varepsilon(x) f(x,y)$ by doing the $x-$ integral first. Then use the dominated convergence theorem to prove Eq. (1.1) by passing to the limit, $\varepsilon \downarrow 0$.