Math 240C Homework Problem List for S2018

0.1 Homework C1. Due Friday, April 6, 2018

- Hand in: 1.1 1.2 1.3 1.5 1.6
- Look at: 1.4
Problems to Solve

Exercise 1.1 (Folland Problem 2.62 on p. 80). Rotation invariance of surface measure on $S^{d-1}$.

Exercise 1.2 (Folland Problem 2.64 on p. 80). On the integrability of $|x|^a |\log |x||^b$ for $x$ near 0 and $x$ near $\infty$ in $\mathbb{R}^n$.

Exercise 1.3. Show, using Problem 1.1 that
$$\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega) = \frac{1}{d} \delta_{ij} \sigma(S^{d-1}).$$

Hint: show $\int_{S^{d-1}} \omega_i^2 d\sigma(\omega)$ is independent of $i$ and therefore
$$\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d} \sum_{j=1}^{d} \int_{S^{d-1}} \omega_j^2 d\sigma(\omega).$$

Exercise 1.4. Let
$$f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Show $f \in C^\infty(\mathbb{R}, [0, 1])$. Hints: you might start by first showing $\lim_{t \downarrow 0} f^{(n)}(t) = 0$ for all $n \in \mathbb{N}_0$.

Exercise 1.5. If $f \in L^1_{\text{loc}}(\mathbb{R}^d, m)$ and $\varphi \in C^1_c(\mathbb{R}^d)$, then $f * \varphi \in C^1(\mathbb{R}^d)$ and $\partial_i (f * \varphi) = f * \partial_i \varphi$. Moreover if $\varphi \in C^\infty_c(\mathbb{R}^d)$ then $f * \varphi \in C^\infty(\mathbb{R}^d)$.

Exercise 1.6 (Integration by Parts). Suppose that $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to f(x, y) \in \mathbb{C}$ and $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1} \to g(x, y) \in \mathbb{C}$ are measurable functions such that for each fixed $y \in \mathbb{R}^d$, $x \to f(x, y)$ and $x \to g(x, y)$ are continuously differentiable. Also assume $f \cdot g, \partial_x f \cdot g$ and $f \cdot \partial_x g$ are integrable relative to Lebesgue measure on $\mathbb{R} \times \mathbb{R}^{d-1}$, where $\partial_x f(x, y) := \frac{d}{dt} f(x + t, y)|_{t=0}$. Show
$$\int_{\mathbb{R} \times \mathbb{R}^{d-1}} \partial_x f(x, y) \cdot g(x, y) dx dy = -\int_{\mathbb{R} \times \mathbb{R}^{d-1}} f(x, y) \cdot \partial_x g(x, y) dx dy. \quad (1.1)$$

Hints: Let $\psi \in C^\infty_c(\mathbb{R})$ be a function which is 1 in a neighborhood of 0 $\in \mathbb{R}$ and set $\psi_\varepsilon(x) = \psi(\varepsilon x)$. First verify Eq. (1.1) with $f(x, y)$ replaced by $\psi_\varepsilon(x) f(x, y)$ by doing the $x$-integral first. Then use the dominated convergence theorem to prove Eq. (1.1) by passing to the limit, $\varepsilon \downarrow 0$. 