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Homework #6 (Spring 2018)

For this last homework assignment:
Hand in Exercises 1.2 and 1.3.
Look at Exercise 1.1.

The next elementary theorem (referred to as the bounded linear transformation theorem, or B.L.T. theorem for short) is often useful when constructing bounded linear transformations.

Theorem 1.1 (B. L. T. Theorem). *Suppose that Z is a normed space, X is a Banach space, and $\mathcal{S} \subset Z$ is a dense linear subspace of Z. If $T : \mathcal{S} \to X$ is a bounded linear transformation (i.e. there exists $C < \infty$ such that $\|Tz\| \leq C \|z\|$ for all $z \in \mathcal{S}$), then T has a unique extension to an element $\bar{T} \in L(Z, X)$ and this extension still satisfies*

$$\left\|\bar{T}z\right\| \leq C \|z\| \text{ for all } z \in \bar{\mathcal{S}}.$$

Exercise 1.1. Prove the B.L.T. Theorem 1.1.

Exercise 1.2 (Dini's Theorem). Let X be a compact topological space and $f_n : X \to [0, \infty)$ be a sequence of continuous functions such that $f_n(x) \downarrow 0$ as $n \to \infty$ for each $x \in X$. Show that in fact $f_n \downarrow 0$ uniformly in x, i.e. $\sup_{x \in X} f_n(x) \downarrow 0$ as $n \to \infty$.[1]

Hint: Given $\varepsilon > 0$, consider the open sets $V_n := \{x \in X : f_n(x) < \varepsilon\}$.

Theorem 1.2 (Riesz Markov Theorem for an Interval). *Let $X = [0, 1]$ and $\lambda \in C(X)^*$ be a positive linear functional. Then there exists a unique Borel measure, μ, on $\mathcal{B}_X$ such that $\lambda(f) = \mu(f)$ for all $f \in C(X, \mathbb{R})$ where*

$$\mu(f) := \int_X f d\mu. \tag{1.1}$$

The following notations will be used in Exercise 1.3 below where you are asked prove the existence part of the Riesz Markov Theorem 1.2 on $[0, 1]$.

[1] More generally, if $g_n, g : X \to \mathbb{R}$ are continuous functions such that $g_n(x) \downarrow g(x)$ as $n \to \infty$ for each $x \in X$, then $g_n(x) \to g(x)$ uniformly in x. Indeed, apply what you have proved to $f_n := g_n - g$.

Notation 1.3 *For $0 \leq a \leq b \leq 1$, let*

$$\nu([a, b]) := \inf\left\{\lambda(f) : 1_{[a,b]} \leq f \in C(X, \mathbb{R})\right\} \tag{1.2}$$

and

$$F(b) := \nu([0, b]) := \inf\left\{\lambda(f) : 1_{[0,b]} \leq f \in C(X, \mathbb{R})\right\}. \tag{1.3}$$

Notation 1.4 *For $0 \leq \alpha < a < b < \beta \leq 1$, let $\chi_{\alpha,a,b,\beta} \in C([0, 1], [0, 1])$ be the piecewise linear function on $[0, 1]$ which is 0 on $[0, \alpha]$, linearly interpolates from 0 to 1 on $[\alpha, a]$, is 1 on $[a, b]$, linearly interpolates from 1 to 0 on $[b, \beta]$, and is 0 again on $[\beta, 1]$. Also for $0 \leq b \leq \beta \leq 1$, let $\theta_{b,\beta} \in C([0, 1], [0, 1])$ be the piecewise linear function on $[0, 1]$ which is 1 on $[0, b]$, linearly interpolates from 1 to 0 on $[b, \beta]$, and is 0 again on $[\beta, 1]$, see Figure 1.1.*

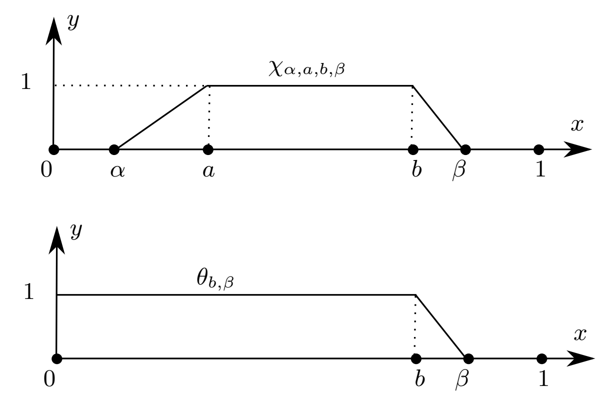

Fig. 1.1. The graphs of smooth approximations to $1_{[a,b]}$ and $1_{[0,b]}$ as continuous functions on $[0, 1]$.

Exercise 1.3 (Riesz Markov Theorem for an Interval). Show there exists a finite Borel measure, μ, on $(X = [0, 1], \mathcal{B} = \mathcal{B}_X)$ satisfying Eq. (1.1) of Theorem 1.2. One way to prove this result is to prove the results listed below.

1. If $0 \leq a \leq b \leq 1$, show $\nu([a, b]) = \lim_{n\to\infty} \lambda(\varphi_n)$ for any sequence, $\{\varphi_n\}_{n=1}^{\infty} \subset C(X, [0, 1])$, such that $\varphi_n(x) \downarrow 1_{[a,b]}(x)$ for all $x \in [0, 1]$. **Suggestions:** given $1_{[a,b]} \leq f \in C(X, [0, 1])$ notice that

a) $\nu([a,b]) \le \lambda(\varphi_n) \le \lambda(\varphi_n \vee f)$ where $\varphi_n \vee f := \max(\varphi_n, f)$, and
b) $\varphi_n \vee f \downarrow f$ uniformly on $[0,1]$ by Dini's theorem.

2. Show $F(b) = \nu([0,b])$ is right continuous in b. **Hint:** if $\{b_n\} \subset (0,1]$ is a strictly decreasing sequence such that $b_n \downarrow b$, then

$$F(b) \le F(b+) \le F(b_n) \le \lambda\left(\theta_{b_n, b_{n-1}}\right). \tag{1.4}$$

Let μ be the unique Borel measure on $[0,1]$ such that $\mu([0,b]) = F(b)$ for all $b \in [0,1]$. The goal is to show that this measure μ satisfies Eq. (1.1).

3. Show $\nu([a,b]) \le \mu([a,b])$ for all $0 \le a \le b \le 1$. **Hints**:
a) if $a = 0$ there is nothing to prove so assume that $0 < a \le b \le 1$.
b) Choose $\{a_n\}_{n=1}^{\infty} \subset (0,a)$ so that a_n strictly increases to a as $n \to \infty$ and let $\varphi_n := \theta_{a_n,a}$ and $\psi_n := \chi_{a_n,a,b,\left(b+\frac{1}{n}\right)\wedge 1}$ and observe that $\varphi_n + \psi_n = \theta_{b,\left(b+\frac{1}{n}\right)\wedge 1}$ and hence

$$F(a_n) + \lambda(\psi_n) \le \lambda(\varphi_n) + \lambda(\psi_n) = \lambda\left(\theta_{b,\left(b+\frac{1}{n}\right)\wedge 1}\right). \tag{1.5}$$

c) Pass to the limit as $n \to \infty$ in the previous inequality.

4. Suppose that $f \in C(X, [0,\infty))$ and $\pi = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1\}$ is a partition of $[0,1]$. Let

$$c_i := \max\{f(t) : t \in [t_{i-1}, t_i]\} \text{ for } 1 \le i \le n$$

and set

$$f_\pi = c_1 1_{[0,t_1]} + c_2 1_{(t_1,t_2]} + \cdots + c_n 1_{(t_{n-1},t_n]}.$$

Show

$$\lambda(f) \le \sum_{i=1}^{n} c_i \nu([t_{i-1}, t_i]) \le \sum_{i=1}^{n} c_i \mu([t_{i-1}, t_i]). \tag{1.6}$$

Hint: If $f_i \in C(X, [0,1])$ satisfy $1_{[t_{i-1},t_i]} \le f_i$ for $1 \le i \le n$, then

$$f \le f_\pi \le \sum_{i=1}^{n} c_i f_i. \tag{1.7}$$

5. Recall that

$$\sum_{x \in [0,1]} \mu(\{x\}) \le \mu([0,1]) = F(1) = \lambda(1) < \infty$$

and hence if $E := \{x \in X : \mu(\{x\}) > 0\}$, then E is at most countable. We now suppose that all partitions, π, we use have now been chosen so that $t_j \notin E$ for $0 < j < n$. Under this assumption, show Eq. (1.6) implies

$$\lambda(f) \le \int_X f_\pi d\mu = \mu(f_\pi).$$

Since $f_\pi \to f$ boundedly (in fact uniformly) as

$$\text{mesh}(\pi) := \max\{|t_i - t_{i-1}| : 1 \le i \le n\} \to 0,$$

conclude that $\lambda(f) \le \mu(f)$.

6. Using $\lambda(1) = \mu(1)$, show $\lambda(f) \le \mu(f)$ for all $f \in C(X, \mathbb{R})$ then apply this result with f replaced by $-f$ to complete the proof.