Math 240 Qual (S2012)

Instructions: Clearly explain and justify your answers. You may cite theorems from the text, notes, or class as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem – this is allowed even if you were unable to prove the previous results. Make sure to state the results that you are using and be sure to verify their hypotheses. All problems have **equal** value. You should do all 8 problems on this test.

Notation: Let m denote Lebesgue measure on $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B} = \mathcal{B}_{\mathbb{R}}$ is the Borel σ – algebra on $\mathbb{R}$. We will write dx for $dm(x)$,

$$\langle f|g\rangle := \int_{\mathbb{R}} f(x)\,\bar{g}(x)\,dx,$$

and let $\|f\|_p$ denote the p – norm of f for all $1 \le p \le \infty$.

Problem 1. Let A and B be two subsets of a topological space (X, τ). Answer the following true or false. If your answer is true prove it and if it is false give a counter example. [In this problem, $\bar{A}$ and A^o denotes the closure and the interior of A respectively.]

1. $\overline{A \cap B} = \bar{A} \cap \bar{B}$.
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$.
3. $(A \cup B)^o = A^o \cup B^o$.

Problem 2. Let $f \in L^1([0,1], m)$ and

$$F(x) := \int_0^1 [\min(x, y) - xy]\, f(y)\, dy.$$

In this problem you are to prove $F''(x) = -f(x)$ in the following sense.

1. Show F is absolutely continuous.
2. Show F' is equal m - a.e. to an absolutely continuous function, G, on $[0,1]$.
3. Show $G' = -f$ a.e.

Problem 3. Suppose $f \in L^1(\mathbb{R}, m)$ and $a := \int_{\mathbb{R}} f(x)\, dx$. Find, with justification, the values of following limits in terms of a.

1. $\lim_{n\to\infty} \int_{\mathbb{R}} e^{ix^2/n} f(x)\, dx$.
2. $\lim_{n\to\infty} \int_{\mathbb{R}} e^{inx} f(x)\, dx$.
3. $\lim_{n\to\infty} \int_{\mathbb{R}} e^{i\pi x^2/n} f(x + \sqrt{n})\, dx$.

Problem 4. Let $1 \le p < \infty$ and for any $\lambda \ne 0$ let T_λ be the linear map from $L^p(\mathbb{R}, m)$ to $L^p(\mathbb{R}, m)$ defined by

$$(T_\lambda f)(x) = f(\lambda x) \text{ for } x \in \mathbb{R} \text{ and } f \in L^p(\mathbb{R}, m).$$

1. Find the operator norm, $\|T_\lambda\|_{op}$, of T_λ for all $\lambda \ne 0$.
2. Explain why $\lim_{\lambda\to 1} \|T_\lambda f - f\|_p = 0$ for all $f \in C_c(\mathbb{R})$.
3. Show $T_\lambda \xrightarrow{s} I$ as $\lambda \to 1$, i.e. $\lim_{\lambda\to 1} \|T_\lambda f - f\|_p = 0$ for all $f \in L^p(\mathbb{R}, m)$.

Problem 5. Let $g : \mathbb{R} \to [0, \infty)$ be a measurable function, $g \in L^1(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$, and let ν be the measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ defined by;

$$\nu(A) = \int_A g(x)\, dm(x) \text{ for all } A \in \mathcal{B}_{\mathbb{R}}. \tag{1.1}$$

Find the Lebesgue decomposition of m relative to ν. In more detail, find a measurable function $\rho : \mathbb{R} \to [0, \infty)$ and a positive measure α on $\mathcal{B}_{\mathbb{R}}$ such that

$$dm = \rho d\nu + d\alpha$$

where α is singular relative to ν. [**Hint:** If you are having trouble with this problem, try the special case where $g(x) > 0$ for $m-$ a.e. x where you should find $\alpha = 0$.]

Problem 6. Let H be a separable Hilbert space, $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H and $\{u_n\}_{n=1}^{\infty}$ be an orthonormal subset of H such that $\delta_n := e_n - u_n$ satisfies

$$\sum_{n=1}^{\infty} \|\delta_n\|_H^2 = \alpha < 1. \tag{1.2}$$

Show $\{u_n\}_{n=1}^{\infty}$ is also an orthonormal basis for H. [**Hint:** For $x \in H$ such that $\langle x|u_n\rangle_H = 0$ for all n, use the given assumptions to estimate $\|x\|^2$.]

Problem 7. Let $\Gamma \subset \{0, 1, 2, \dots\}$ and μ be a complex measure on $\mathcal{B}_{[-\pi/2,\pi/2]}$ such that

$$\int_{[-\pi/2,\pi/2]} \sin^n(x) d\mu = 0 \text{ for } n \in \Gamma \tag{1.3}$$

with the understanding that $\sin^0(x) = 1$ for all x.

1. Show $\mu \equiv 0$ if Eq. (1.3) holds for $\Gamma = \{0, 1, 2, \dots\}$.
2. Find all complex measures μ such that Eq. (1.3) holds for $\Gamma = \mathbb{N} = \{1, 2, \dots\}$.

Problem 8. Recall that the Riemann-Lebesgue lemma implies that

$$\lim_{n\to\infty} \int_0^1 f(x) e^{inx} dx = 0$$

for all $f \in L^1([0,1], dm)$. Show that there is no rate of convergence independent of $f \in L^1([0,1], dm)$ for the above limit. That is show that there is no function $\varphi : \mathbb{N} \to (0, \infty)$ such that

i) $\lim_{n\to\infty} \varphi(n) = \infty$ and
ii) for all $f \in L^1([0,1], dm)$ there is a constant $C = C(f) < \infty$ such that

$$\left| \int_0^1 f(x) e^{inx} dx \right| \le C(f) \frac{1}{\varphi(n)} \quad \forall n \in \mathbb{N}. \tag{1.4}$$

Hint: consider the linear functionals

$$\Lambda_n(f) := \varphi(n) \int_0^1 f(x) e^{inx} dx.$$
