240C Supplements

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The Polar Decomposition of Lebesgue Measure

Let
\[ S^{d-1} = \{ x \in \mathbb{R}^d : |x|^2 := \sum_{i=1}^{d} x_i^2 = 1 \} \]
and
\[ B' := \{ x \in \mathbb{R}^d : 0 < \|x\| \leq 1 \} \]
be the unit sphere and 0-deleted “closed” ball in \( \mathbb{R}^d \) and let \( B_{S^{d-1}} \) and \( B_{E'} \) be the Borel \( \sigma \)-algebras on these metric spaces. We further equip \( (0, \infty) \times S^{d-1} \) with product \( \sigma \)-algebra \( B_{(0, \infty)} \otimes B_{S^{d-1}} \) which is also the Borel \( \sigma \)-algebra on \( (0, \infty) \times S^{d-1} \) thought of as a product of two metric spaces. The maps \( \Phi : \mathbb{R}^d \setminus \{0\} \to (0, \infty) \times S^{d-1} \) and \( \psi : B' \to S^{d-1} \) defined by
\[
\Phi(x) := (|x|, |x|^{-1} x) \text{ for all } x \in \mathbb{R}^d \setminus \{0\} \text{ and }
\psi(x) = |x|^{-1} x \text{ for all } x \in B',
\]
are both continuous and hence measurable. Similarly the inverse map, \( \Phi^{-1} : (0, \infty) \times S^{d-1} \to \mathbb{R}^d \setminus \{0\} \), is given by \( \Phi^{-1}(r, \omega) = r\omega \) which is continuous and therefore also measurable.

For \( E \in B_{S^{d-1}} \) and \( a > 0 \), let
\[
E_a := \{ r\omega : r \in (0, a] \text{ and } \omega \in E \} = \Phi^{-1}((0, a] \times E) \in B_{\mathbb{R}^d}.
\]
Further observe that \( E_1 = \psi^{-1}(E) \in B_{E'} \subset B_{\mathbb{R}^d} \) and for \( a > 0 \), \( E_a = aE_1 \).

**Definition 1.1.** The *surface measure*, \( \sigma \), on \( S^{d-1} \) is defined to be \( \sigma = d \cdot (\psi \cdot m) \), i.e.
\[
\sigma(E) := d \cdot m(E_1) \text{ for all } E \in B_{S^{d-1}}.
\]

Let us now explain the intuition behind Definition 1.1. If \( E \subset S^{d-1} \) is a set and \( \varepsilon > 0 \) is a small number, then the volume of
\[
(1, 1 + \varepsilon) \cdot E = \{ r\omega : r \in (1, 1 + \varepsilon) \text{ and } \omega \in E \}
\]
should be approximately given by \( m((1, 1 + \varepsilon) \cdot E) \approx \sigma(E) \varepsilon \), see Figure 1.1 below.

On the other hand
\[
m((1, 1 + \varepsilon)E) = m(E_{1+\varepsilon} \setminus E_1) = [(1 + \varepsilon)^d - 1] \cdot m(E_1) .
\]

Therefore we expect the area of \( E \) should be given by
\[
\sigma(E) = \lim_{\varepsilon \downarrow 0} \frac{[(1 + \varepsilon)^d - 1]}{\varepsilon} m(E_1) = d \cdot m(E_1).
\]

The following theorem is an abstract version of integration in polar coordinates.

**Theorem 1.2 (Polar decomposition of \( m \)).** Let \( \rho_d \) be the measure on \( \mathcal{B}_{(0, \infty)} \) defined by \( d\rho_d(r) = r^{d-1}dr \), i.e.
\[
\rho(J) = \int_J r^{d-1}dr \quad \forall \, J \in \mathcal{B}_{(0, \infty)},
\]
then \( \Phi_* m = \rho \otimes \sigma \) on \( \mathcal{B}_{(0, \infty)} \otimes \mathcal{B}_{S^{d-1}} \).

**Proof.** Let \( \mathcal{E} \) be the \( \pi \)-system in \( \mathcal{B}_{(0, \infty)} \otimes \mathcal{B}_{S^{d-1}} \) consisting of sets of the form \( A = (a, b] \times E \in \mathcal{B}_{(0, \infty)} \otimes \mathcal{B}_{S^{d-1}} \) with \( 0 < a < b < \infty \) and \( E \in \mathcal{B}_{S^{d-1}} \). For such an \( A \in \mathcal{E} \) we have
\[
\Phi^{-1}(A) = \{ r\omega : r \in (a, b] \text{ and } \omega \in E \} = E_b \setminus E_a = bE_1 \setminus aE_1.
\]
Therefore by the basic scaling properties of \( m \) and the fundamental theorem of calculus,
\[
(\Phi_* m)(A) = m(bE_1 \setminus aE_1) = m(bE_1) - m(aE_1)
\]
\[
= b^d m(E_1) - a^d m(E_1) = d \cdot (m(E_1) \int_a^b r^{d-1} dr) \tag{1.2}
\]
\[
= \rho ((a, b]) \sigma (E) = (\rho \otimes \sigma) (A) \tag{1.3}
\]
Since \((\Phi_* m)(A) = (\rho \otimes \sigma)(A)\) for all \( A \in \mathcal{E} \), we may apply the multiplicative system theorem\(^1\) in the form of Proposition ?? to conclude that \( \Phi_* m = \rho \otimes \sigma \) on \( B_{(0, \infty)} \otimes B_{S^{d-1}} \).

**Corollary 1.3 (Polar Coordinates).** If \( f : \mathbb{R}^d \to [0, \infty) \) is a \((B_{\mathbb{R}^d}, \mathcal{B})\)-measurable function then
\[
\int_{\mathbb{R}^d} f(x) dm(x) = \int_{(0, \infty) \times S^{d-1}} f(r \omega) r^{d-1} dr d\sigma(\omega). \tag{1.4}
\]

In particular if \( f : \mathbb{R}^d \to \mathbb{R}_+ \) is measurable then
\[
\int_{\mathbb{R}^d} f(|x|) dx = \sigma (S^{d-1}) \int_0^\infty f(r) r^{d-1} dr = \int_0^\infty f(r) dV(r) \tag{1.5}
\]
where
\[
V(r) = m(B(0, r)) = r^d m(B(0, 1)) = d^{-1} r^d \sigma (S^{d-1}).
\]

In Example ??, Exercise ??, and Proposition ?? below, we will use the general change of variables Theorem ?? to give a explicit description for the surface integrals relative to \( \sigma \).

**Proof.** Equation (1.4) is a direct consequence of the abstract change of variables theorem (Exercise ??), Theorem 1.2, and Tonelli’s Theorem ??, Indeed we have,
\[
\int_{\mathbb{R}^d} f dm = \int_{\mathbb{R}^d \setminus \{0\}} (f \circ \Phi^{-1}) \circ \Phi dm = \int_{(0, \infty) \times S^{d-1}} (f \circ \Phi^{-1}) d(\Phi_* m)
\]
\[
= \int_{(0, \infty) \times S^{d-1}} (f \circ \Phi^{-1}) d [\rho \otimes \sigma] = \int_{(0, \infty) \times S^{d-1}} f(r \omega) \rho (dr) \sigma (d\omega)
\]
\[
= \int_{(0, \infty) \times S^{d-1}} f(r \omega) r^{d-1} dr d\sigma(\omega).
\]
Equation (1.5) is a special case of Eq. (1.4).

\(^1\) Or you could use the \( \pi \)-\( \lambda \) theorem.

**Example 1.4 (\( \sigma (S^1) = 2\pi \)).** Let \( E = \{(\cos \theta, \sin \theta) \in S^1 \subset \mathbb{R}^2 : 0 \le \theta \le \pi \} \), then \( E_1 \) is the upper half of closed unit disk centered at \( 0 \) in \( \mathbb{R}^2 \). Therefore,
\[
m^2 (E_1) = \int_{E_1} dx = \int_{-1}^1 dx \int_0^{\sqrt{1-x^2}} dy = \int_{-1}^1 \sqrt{1-x^2} dx.
\]
Letting \( x = \sin \theta \) we find,
\[
\sigma (E) = 2m^2 (E_1) = 2 \int_{-\pi/2}^{\pi/2} \cos \theta \cdot \cos \theta d\theta
\]
\[
= 2 \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} [1 + \cos 2\theta] d\theta = \pi.
\]
Therefore \( \sigma (S^1) = 2\sigma (E) = 2\pi \) – the circumference of a circle of radius \( 1 \) as to be expected.

**Lemma 1.5.** If \( a > 0 \) and \( d \in \mathbb{N} \), then
\[
I_d (a) := \int_{\mathbb{R}^d} e^{-a|x|^2} dm(x) = (\pi/a)^{d/2}.
\]
**Proof.** Using Tonelli’s theorem and induction,
\[
I_d (a) = \int_{\mathbb{R}^{d-1} \times \mathbb{R}} e^{-a|y|^2} e^{-ar^2} m_{d-1}(dy) \ d\theta
\]
\[
= I_{d-1} (a) I_1 (a) = I_1^d (a). \tag{1.6}
\]
So it suffices to compute:
\[
I_2 (a) = \int_{\mathbb{R}^2} e^{-a|x|^2} dm(x) = \int_{\mathbb{R}^2 \setminus \{0\}} e^{-a(x_1^2 + x_2^2)} dx_1 dx_2.
\]
Using polar coordinates, see Eq. (1.4), we find,
\[
I_2 (a) = \int_{(0, \infty) \times S^1} e^{-a|r|^2} r \ dr d\sigma (\omega) = \sigma (S^1) \cdot \int_0^\infty e^{-ar^2} dr
\]
\[
= 2\pi \lim_{M \to \infty} \int_0^M e^{-ar^2} dr = 2\pi \lim_{M \to \infty} \frac{e^{-ar^2}}{-2a} \bigg|_0^M = \frac{2\pi}{2a} = \pi/a.
\]
This shows that \( I_2 (a) = \pi/a \) and the result now follows from Eq. (1.6).
Corollary 1.6. The surface area $\sigma(S^{d-1})$ of the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is
\[
\sigma(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{1.7}
\]
where $\Gamma$ is the gamma function is as in Example ?? and ??.

Proof. Using Corollary 1.3 we find
\[
I_d(1) = \int_0^\infty dr \, r^{d-1} e^{-r^2} \int_{S^{d-1}} d\sigma = \sigma(S^{d-1}) \int_0^\infty r^{d-1} e^{-r^2} dr.
\]
Making the making the change of variables $u = r^2$ so that $r = u^{1/2}$ and $dr = \frac{1}{2}u^{-1/2}du$ we find
\[
\frac{I_d(1)}{\sigma(S^{d-1})} = \int_0^\infty u^{d-1} e^{-u} \frac{1}{2} u^{-1/2}du = \frac{1}{2} \int_0^\infty u^{d-1} e^{-u}du = \frac{1}{2} \Gamma(d/2). \tag{1.8}
\]
Solving this equation for $\sigma(S^{d-1})$ while making use of Lemma 1.5 gives Eq. 1.7.

Exercise 1.1 (Folland Problem 2.62 on p. 80.). Rotation invariance of surface measure on $S^{n-1}$.

Exercise 1.2 (Folland Problem 2.64 on p. 80.). On the integrability of $|x|^a |\log|x||^b$ for $x$ near 0 and $x$ near $\infty$ in $\mathbb{R}^n$.

Exercise 1.3. Show, using Problem 1.1 that
\[
\int_{S^{d-1}} \omega_i \omega_j d\sigma(\omega) = \frac{1}{d} \delta_{ij} \sigma(S^{d-1}).
\]

Hint: show $\int_{S^{d-1}} \omega_i^2 d\sigma(\omega)$ is independent of $i$ and therefore
\[
\int_{S^{d-1}} \omega_i^2 d\sigma(\omega) = \frac{1}{d} \sum_{j=1}^d \int_{S^{d-1}} \omega_j^2 d\sigma(\omega).
\]

Proposition 1.7. Let $d \in \mathbb{N}$,
\[
\mathbb{R}^d_+ \coloneqq \{ x \in \mathbb{R}^d : x_i \geq 0 \text{ for } 1 \leq i \leq d \}, \quad Z^d_+ = \mathbb{Z}^d \cap \mathbb{R}^d_+,
\]
and $f(r) \geq 0$ is a continuous decreasing (i.e. non-increasing) function of $r \geq 0$. With this notation we have
\[
\sum_{k \in Z^d_+} f(\|k\|) < \infty \iff \int_0^\infty f(r) r^{d-1} dr < \infty.
\]

Proof. Let us set $f(r) = f(0)$ for $r \leq 0$ and let $Q := (0,1]^d$ and for $k \in Z^d_+$, let $Q_k := k + Q$ be the translate of $Q$ by $k$. For any $x = k + y \in Q_k$ we have
\[
\|k\| \leq \|x\| \text{ and } \|x\| \leq \|k\| + \|y\| \leq \|k\| + \sqrt{d},
\]
i.e.
\[
\|x\| - \sqrt{d} \leq \|k\| \leq \|x\| \text{ for } x \in Q_k.
\]
Thus it follows that
\[
\int_{Q_k} \|x\| - \sqrt{d} \geq \int_{Q_k} f(\|k\|) \geq \int_{Q_k} f(\|x\|) \text{ for } x \in Q_k.
\]
Thus if let
\[
F(x) := \sum_{k \in Z^d_+} f(\|k\|) 1_{Q_k}(x)
\]
we have shown
\[
\int_{Q_k} f(\|x\| - \sqrt{d}) \geq F(x) \geq f(\|x\|) \text{ for } x \in \mathbb{R}^d.
\]
Recalling that
\[
\int_{\mathbb{R}^d_+} f(\|x\|) dm(x) = c_d \int_0^\infty f(r) r^{d-1} dr
\]
for some constant $c_d < \infty$, we may integrate Eq. (1.9) over $\mathbb{R}^d_+$ to find,
\[
c_d \int_0^\infty f(r - \sqrt{d}) r^{d-1} dr \geq \sum_{k \in Z^d_+} f(\|k\|) \geq c_d \int_0^\infty f(r) r^{d-1} dr.
\]
Since
\[
\int_0^\infty f(r - \sqrt{d}) r^{d-1} dr = \int_{-\sqrt{d}}^{\infty} f(s + \sqrt{d}) \frac{d-1}{d} ds \leq f(0) \int_{-\sqrt{d}}^{\sqrt{d}} (s + \sqrt{d})^{-d} ds + \int_{\sqrt{d}}^\infty f(s) (s + \sqrt{d})^{-d} ds \leq f(0) \cdot C(d) + 2^{d-1} \int_{\sqrt{d}}^{\infty} f(s) s^{d-1} ds
\]
and we have shown,
\[
c_d f(0) \cdot C(d) + c_d 2^{d-1} \int_0^\infty f(s) s^{d-1} ds \geq \sum_{k \in Z^d_+} f(\|k\|) \geq c_d \int_0^\infty f(r) r^{d-1} dr
\]
from which the result easily follows.
Corollary 1.8. If $d \in \mathbb{N}$ and $f(r) \geq 0$ is a continuous decreasing (i.e. non-increasing) function of $r \geq 0$, then

$$
\sum_{k \in \mathbb{Z}^d} f(\|k\|) < \infty \iff \int_0^\infty f(r) r^{d-1} dr < \infty.
$$

Proof. For $\varepsilon \in \{\pm 1\}^d$, let $\mathbb{Z}_\varepsilon^d = \{k \in \mathbb{Z}^d : \varepsilon_i k_i \geq 0 \text{ for } 1 \leq i \leq d\}$. Then

$$
\sum_{k \in \mathbb{Z}_\varepsilon^d} f(\|k\|) = \sum_{k \in \mathbb{Z}^d} f(\|k\|) \text{ for all } \varepsilon \in \{\pm 1\}^d
$$

and since $\mathbb{Z}_\varepsilon^d \subset \mathbb{Z}^d \subset \bigcup_\varepsilon \mathbb{Z}_\varepsilon^d$ it follows that

$$
\sum_{k \in \mathbb{Z}^d} f(\|k\|) < \infty \iff \sum_{k \in \mathbb{Z}_\varepsilon^d} f(\|k\|) < \infty \iff \int_0^\infty f(r) r^{d-1} dr < \infty.
$$
Metric-Measure Space Regularity Results

This section is a self study guide to the “approximating” Borel sets in a metric space by closed and open subsets of the metric space. We will see similar results in more general topological spaces later in the book. [See Section ?? and also subsection ?? for related results.] We begin with some basic properties of metric spaces. Throughout this section we will assume that \((X, \rho)\) is a metric space and \(B_X\) denotes the Borel \(\sigma\)-algebra on \(X\).

2.1 Metric space results

Lemma 2.1. For any non empty subset \(A \subset X\), let \(\rho_A(x) := \inf \{\rho(x,a) | a \in A\}\), then
\[|\rho_A(x) - \rho_A(y)| \leq \rho(x,y) \quad \forall x, y \in X\] (2.1)
which shows \(\rho_A : X \to [0, \infty)\) is continuous.

Proof. Let \(a \in A\) and \(x, y \in X\), then
\[\rho_A(x) \leq \rho(x, a) \leq \rho(x, y) + \rho(y, a)\]
Take the infimum over \(a\) in the above equation shows that
\[\rho_A(x) \leq \rho(x, y) + \rho_A(y) \quad \forall x, y \in X\]
Therefore, \(\rho_A(x) - \rho_A(y) \leq \rho(x, y)\) and by interchanging \(x\) and \(y\) we also have that \(\rho_A(y) - \rho_A(x) \leq \rho(x, y)\) which implies Eq. (2.1).

Corollary 2.2. The function \(\rho\) satisfies,
\[|\rho(x, y) - \rho(x', y')| \leq \rho(y, y') + \rho(x, x')\]
In particular \(\rho : X \times X \to [0, \infty)\) is continuous.

Proof. By Lemma 2.1 for single point sets and the triangle inequality for the absolute value of real numbers,
\[|\rho(x, y) - \rho(x', y')| \leq |\rho(x, y) - \rho(x, y')| + |\rho(x, y') - \rho(x', y')| \leq \rho(y, y') + \rho(x, x')\]

Corollary 2.3. Given any set \(A \subset X\) and \(\varepsilon > 0\), then
\[A_\varepsilon := \{\rho_A < \varepsilon\} := \{x \in X : \rho_A(x) < \varepsilon\}\]
is an open set containing \(A\) and \(A_\varepsilon \downarrow \bar{A}\) as \(\varepsilon \downarrow 0\) where \(\bar{A}\) is the closure of \(A\). Similarly,
\[F_\varepsilon := \{\rho_A \geq \varepsilon\} = \{x \in X : \rho_A(x) \geq \varepsilon\}\]
is a closed set and \(F_\varepsilon \uparrow (A^c)^o\) as \(\varepsilon \downarrow 0\) where \((A^c)^o\) is the interior of \(A^c := X \setminus A\).

Proof. Because of the continuity of \(\rho_A\) and the facts that \((-\infty, \varepsilon)\) open in \(\mathbb{R}\) and \([\varepsilon, \infty)\) is closed in \(\mathbb{R}\), it follows that \(A_\varepsilon = \rho_A^{-1}((-\infty, \varepsilon))\) is open and \(F_\varepsilon = \rho_A^{-1}([\varepsilon, \infty))\) is closed. We have \(x \in \cap_{\varepsilon > 0} A_\varepsilon\) iff \(\rho_A(x) < \varepsilon\) for all \(\varepsilon > 0\) iff \(\rho_A(x) = 0\) and hence
\[A \subset \{\rho_A = 0\} = \cap_{\varepsilon > 0} A_\varepsilon\]
Since \(\{\rho_A = 0\}\) is closed it follows that \(\bar{A} \subset \{\rho_A = 0\}\). Conversely if \(x \in \{\rho_A = 0\}\) then there exists \(\{x_n\} \subset A\) such that \(\lim_{n \to \infty} \rho(x, x_n) = 0\), i.e. \(x_n \to x\) and therefore \(x \in \bar{A}\).
To finish the proof observe that
\[\bigcup_{\varepsilon > 0} F_\varepsilon^c = \cap_{\varepsilon > 0} F_\varepsilon^c = \cap_{\varepsilon > 0} \{\rho_A < \varepsilon\} = \bar{A}\]
and therefore
\[\bigcup_{\varepsilon > 0} F_\varepsilon = \bar{A}^c = (A^c)^o\]

Lemma 2.4 (Urysohn’s Lemma for Metric Spaces). Let \((X, d)\) be a metric space and suppose that \(A\) and \(B\) are two disjoint closed subsets of \(X\). Then
\[f(x) = \frac{d_B(x)}{d_A(x) + d_B(x)} \quad \text{for} \quad x \in X\] (2.2)
defines a continuous function, \(f : X \to [0, 1]\), such that \(f(x) = 1\) for \(x \in A\) and \(f(x) = 0\) if \(x \in B\).

Proof. By Lemma 2.1 \(d_A\) and \(d_B\) are continuous functions on \(X\). Since \(A\) and \(B\) are closed, \(d_A(x) > 0\) if \(x \notin A\) and \(d_B(x) > 0\) if \(x \notin B\). Since
A \cap B = \emptyset, d_A (x) + d_B (x) > 0 \text{ for all } x \text{ and } (d_A + d_B)^{-1} \text{ is continuous as well.}

The remaining assertions about $f$ are all easy to verify. ■

Sometimes Urysohn’s lemma will be used in the following form. Suppose $F \subset V \subset X$ with $F$ being closed and $V$ being open, then there exists $f \in C (X, [0, 1])$ such that $f = 1$ on $F$ while $f = 0$ on $V^c$. This of course follows from Lemma \ref{lem:metric-measure} by taking $A = F$ and $B = V^c$.

**Corollary 2.5.** If $A$ and $B$ are two disjoint closed subsets of a metric space, $(X, d)$, then there exists disjoint open subsets $U$ and $V$ of $X$ such that $A \subset U$ and $B \subset V$.

**Proof.** Let $f$ be as in Lemma \ref{lem:metric-measure} so that $f \in C (X \to [0, 1])$ such that $f = 1$ on $A$ and $f = 0$ on $B$. Then set $U = \{ f > \frac{1}{2} \}$ and $V = \{ f < 1/2 \}$. ■

We end this subsection with the following simple variant of Proposition \ref{lem:metric-measure}. This proposition shows how to associate a pseudo metric to any measure space.

**Proposition 2.6 (The measure pseudo metric).** Let $(\Omega, \mathcal{B}, \mu)$ be a measure space and define

$$d_{\mu} (A, B) := \mu (A \triangle B) \in [0, \infty) \ \forall A, B \in \mathcal{B}.$$ 

Then $d = d_{\mu}$ satisfies;

1. $d$ is a pseudo metric, i.e. $d (A, B) = d (B, A)$ and $d (A, C) \leq d (A, B) + d (B, C)$ for all $A, B, C \in \mathcal{B}$.
2. $d (A^c, C^c) = d (A, C)$ for all $A, B \in \mathcal{B}$.
3. If $\{A_n\}_{n=1}^\infty, \{B_n\}_{n=1}^\infty \subset \mathcal{B}$, then

$$d (\bigcup_{n=1}^\infty A_n, \bigcap_{n=1}^\infty B_n) \leq \sum_{n=1}^\infty d (A_n, B_n) \quad (2.3)$$

$$d (\bigcap_{n=1}^\infty A_n, \bigcup_{n=1}^\infty B_n) \leq \sum_{n=1}^\infty d (A_n, B_n). \quad (2.4)$$

In summary,

$$\max \{ d (\bigcap_{n=1}^\infty A_n, \bigcap_{n=1}^\infty B_n), d (\bigcup_{n=1}^\infty A_n, \bigcup_{n=1}^\infty B_n) \} \leq \sum_{n=1}^\infty d (A_n, B_n). \quad (2.5)$$

**Proof.** We take each item in turn.

1. The fact that $d$ is a pseudo metric easily follows from the fact that $1_{A \triangle C} = |1_A - 1_C|$ and therefore,

$$d (A, C) = \|1_A - 1_C\|_1.$$ 

2. Item 2 follows from the fact that

$$A^c \triangle C^c = [A^c \cap C] \cup [C^c \cap A] = [C \setminus A] \cup [A \setminus C] = A \triangle C$$

which is also seen via,

$$1_{A^c \triangle C^c} = |1_A - 1_{C^c}| = |1 - 1_A| - |1 - 1_C| = |1_A - 1_C| = 1_{A \triangle C},$$

3. It is a simple exercise to verify,

$$[\bigcup_{n=1}^\infty A_n] \bigtriangleup [\bigcup_{n=1}^\infty B_n] \subset \bigcup_{n=1}^\infty [A_n \bigtriangleup B_n]$$

and hence

$$d (\bigcup_{n=1}^\infty A_n, \bigcup_{n=1}^\infty B_n) = \mu ([\bigcup_{n=1}^\infty A_n] \bigtriangleup [\bigcup_{n=1}^\infty B_n]) \leq \mu (\bigcup_{n=1}^\infty [A_n \bigtriangleup B_n])$$

$$\leq \sum_{n=1}^\infty \mu (A_n \bigtriangleup B_n) = \sum_{n=1}^\infty d (A_n, B_n),$$

which proves Eq. \ref{eq:metric-measure}. Equation \ref{eq:metric-measure} may be proved similarly or by combining item 2. with Eq. \ref{eq:metric-measure} as follows;

$$d (\bigcap_{n=1}^\infty A_n, \bigcap_{n=1}^\infty B_n) = d (\bigcap_{n=1}^\infty A_n)^c, (\bigcap_{n=1}^\infty B_n)^c$$

$$= d (\bigcup_{n=1}^\infty A_n^c, \bigcup_{n=1}^\infty B_n^c) \leq \sum_{n=1}^\infty d (A_n^c, B_n^c) = \sum_{n=1}^\infty d (A_n, B_n).$$

■

## 2.2 Regularity Results for Borel measures on $(X, \mathcal{B}_X)$

**Exercise 2.1.** If $(X, \rho)$ is a metric space and $\mu$ is a finite measure on $(X, \mathcal{B}_X)$, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $\mu (V \setminus F) = \mu (F \setminus V) < \varepsilon$. Here are some suggestions.

1. Let $B_0$ denote those $A \subset X$ such that for all $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $d_{\mu} (F, V) = \mu (V \setminus F) < \varepsilon$.
2. Show $B_0$ contains all closed (or open if you like) sets using Corollary \ref{lem:metric-measure}.
3. Show $B_0$ is a $\sigma$-algebra. [You may find Proposition \ref{prop:metric-measure} to be helpful in this step.]
4. Explain why this proves the result.
Exercise 2.2. Let $(X, \rho)$ be a metric space and $\mu$ be a measure on $(X, \mathcal{B}_X)$. If there exists open sets, $\{V_n\}_{n=1}^\infty$ of $X$ such that $V_n \uparrow X$ and $\mu (V_n) < \infty$ for all $n$, then for all $A \in \mathcal{B}_X$ and $\varepsilon > 0$ there exists a closed set $F$ and open set $V$ such that $F \subset A \subset V$ and $d_\mu(F, V) = \mu(V \setminus F) < \varepsilon$. **Hints:**

1. Show it suffices to prove; for all $\varepsilon > 0$ and $A \in \mathcal{B}_X$, there exists an open set $V \subset X$ such that $A \subset V$ and $\mu (V \setminus A) < \varepsilon$.

2. Now you must verify the assertion above holds. For this, you may find it useful to apply Exercise 2.1 to the measures, $\mu_n : \mathcal{B}_X \to [0, \mu(V_n)]$, defined by $\mu_n(A) := \mu(A \cap V_n)$ for all $A \in \mathcal{B}_X$. The $\varepsilon$ in Exercise 2.1 should be replaced by judiciously chosen small quantities depending on $n$.

**Theorem 2.7.** Suppose that $(X, \rho)$ is a metric space and $\mu$ is a measure on $(X, \mathcal{B}_X)$ such that $\mu (K) < \infty$ whenever $K$ is a compact subset of $X$. If there exists open sets, $\{V_n\}_{n=1}^\infty$, of $X$ such that $V_n \uparrow X$ and $V_n$ is compact for all $n \in \mathbb{N}$, then $C_c (X, \mathbb{C})$ is dense in $L^p (\mu)$ for all $1 \leq p < \infty$.

**Proof.** Suppose that $A \in \mathcal{B}_X$ is a set such that $\mu (A) < \infty$ and let $\varepsilon > 0$ be given. By Exercise 2.2 there exists a closed set $F$ and an open set $V$ such that $F \subset A \subset V$ and $\mu (V \setminus F) < \varepsilon$. [Note that $\mu (F) \leq \mu (A) \leq \mu (V) \leq \mu (A) + \varepsilon < \infty$.] For each $m \in \mathbb{N}$, $K_m := F \cap V_m$ are compact subsets of $F$ such that $K_m \uparrow F$ as $m \uparrow \infty$. By DCT if follows $\lim_{m \to \infty} \mu(V \setminus K_m) = \mu (V \setminus F) < \varepsilon$ and hence there exists a $m \in \mathbb{N}$ so that $\mu (V \setminus K_m) < \varepsilon$. Thus if we let $K := K_m$, then $K$ is compact, $K \subset A \subset V$, and $\mu (V \setminus K) < \varepsilon$. Moreover, since $K \subset X = \bigcup_{n=1}^\infty V_n$, there exists (by compactness) an $n \in \mathbb{N}$ such that $K \subset V_n \subset V$.

We now define $\delta := \rho (K_m, [V_n \cap V]) > 0$ and then define

$$f (x) = \left[1 - \frac{2}{\delta} \rho_K (x)\right]^+$$

for all $x \in X$.

Since

$$\{f > 0\} \subset \left\{1 - \frac{2}{\delta} \rho_K > 0\right\} \subset \left\{\rho_K < \frac{1}{2}\right\} \subset \left\{\rho_K \leq \frac{\delta}{2}\right\} \subset V_n \cap V,$$

it follows that

$$\text{supp}(f) = \{f > 0\} \subset V_n \subset V_n.$$

Thus $f \in C_c (X, [0, 1])$, $f = 1$ on $K$, and $f = 0$ on $V^c$, and hence

$$|f - 1_A| \leq 1_{V \cap V_n \setminus K} \leq 1_{V \setminus K}$$

from which it follows that

$$\|f - 1_A\|_p \leq \|1_{V \setminus K}\|_p \leq \varepsilon^{1/p}.$$ 

As $\varepsilon > 0$ was arbitrary, we have shown $1_A \in C_c (X, \mathbb{C}) L^p (\mu)$ for all $A \in \mathcal{B}_X$ with $\mu (A) < \infty$. This completes the proof since simple functions which are in $L^p (\mu)$ are known to be dense in $L^p (\mu)$. **■**

**Corollary 2.8.** If $X$ is an open subset of $\mathbb{R}^n$ and $\mu$ is a measure on $\mathcal{B}_X$ such that $\mu(K) < \infty$ for all compact subsets, $K \subset U$, then $C_c (X, \mathbb{C})$ is dense in $L^p (\mu)$ for all $1 \leq p < \infty$.

**Proof.** Let $\rho (x, y) := |y - x|$ be the usual Euclidean metric on $X \subset \mathbb{R}^n$ and define

$$V_n := \left\{\rho_{X^c} > \frac{1}{n}\right\} \cap B_\rho (0, n),$$

where

$$B_\rho (0, n) := \{x \in \mathbb{R}^n : \rho (x, 0) = |x| < n\}.$$

Then $V_n$ is an open subset of $X$ such that

$$V_n \subset \left\{\rho_{X^c} \geq \frac{1}{n}\right\} \cap B_\rho (0, n) \subset X.$$

As $V_n$ is closed and bounded it is compact and since $V_n \uparrow X$ as $n \to \infty$, the result now follows by an application of Theorem 2.7. **■**

**Corollary 2.9.** Suppose that $(X, \rho)$ is a metric space with open sets, $\{V_n\}_{n=1}^\infty$ of $X$ such that $V_n \uparrow X$ and $V_n$ is compact for all $n \in \mathbb{N}$ and $\mu$ is a complex measure on $(X, \mathcal{B}_X)$.

If $f_X d\mu = 0$ for all $f \in C_c(X)$, then $\mu = 0$.

**Proof.** If we let $\mu := |\nu|$, then there is a measurable function, $g : X \to S^1 \subset \mathbb{C}$ such that $d\nu = gd\mu$. Since $C_c (X)$ is dense in $L^1 (\mu)$, there exists $f_n \in C_c (X)$ such that $f_n \to \hat{g}$ in $L^1 (\mu)$ as $n \to \infty$. Therefore,

$$0 = \int_X f_n d\nu = \int_X f_n g d\mu \to \int_X \hat{g} d\mu = \int_X d\mu = \mu (X).$$

This shows $\mu (X) = 0$ and hence $\nu = 0$. **■**

**Definition 2.10.** If $\nu$ is a complex measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$, let $\hat{\nu} : \mathbb{R}^n \to \mathbb{C}$ be the **characteristic function of $\nu$** defined by,

$$\hat{\nu} (\lambda) := \int_{\mathbb{R}^n} e^{-i\lambda x} d\nu (x) \quad \forall \lambda \in \mathbb{R}^n.$$

**Corollary 2.11.** Let $\nu$ be a complex measure on $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$. If $\hat{\nu} \equiv 0$, then $\nu = 0$, i.e. the linear map,

$$\text{complex measures on } \mathbb{R}^n \ni \nu \to \hat{\nu} \in \{\text{functions on } \mathbb{R}^n\},$$

is injective.
2.3 Dual Considerations

As in Theorem 2.7 let us suppose for simplicity that \((X, \rho)\) is a metric space such that there exists open sets, \(\{V_n\}_{n=1}^\infty\), of \(X\) such that \(V_n \uparrow X\) and \(V_n\) is compact for all \(n \in \mathbb{N}\). We now suppose that \(\nu\) is a complex measure on \((X, \mathcal{B}_X)\) and for \(f \in C_0(X)\), let

\[\nu(f) := \int_X f d\nu,\]

i.e. we identify \(\nu\) with an element of \(C_0(X)^*\). Our goal is to show \(\|\nu\|_{c_0(X)^*} = |\nu|(X)\). To do this we will use the following simple lemma.

**Lemma 2.14 (Sliding points).** Let \(\varphi : \mathbb{C} \rightarrow D := \{z \in \mathbb{C} : |z| \leq 1\}\) be defined by

\[\varphi(z) = (|z| \vee 1)^{-1} z = \begin{cases} \frac{z}{|z|} & \text{if } |z| \leq 1, \\ \frac{zt}{|t|^2} & \text{if } |z| > 1. \end{cases}\]

Then \(\varphi\) is continuous and satisfies

\[|\varphi(z) - w| \leq |z - w| \quad \forall \ z \in \mathbb{C} \text{ and } w \in S^1,\]

see Figure 2.1.

![Fig. 2.1. Sliding points to the unit circle.](image)

**Proof.** It is easy to verify \(\varphi\) is continuous. If \(w, z \in S^1\) then

\[
\frac{d}{dt} |w - tz|^2 = \frac{d}{dt} \left[1 + t^2 - 2t \text{Re}(\bar{w}z)\right] = 2[t - \text{Re}(\bar{w}z)] > 0 \quad \text{if } t > 1.
\]

This shows \(|w - \varphi(tz)| \geq |w - z|\) for all \(t \geq 1\).

**Theorem 2.15 (Dual of \(C_0(X)\)).** Let \((X, \rho)\) be a metric space such that there exists open sets, \(\{V_n\}_{n=1}^\infty\), of \(X\) such that \(V_n \uparrow X\) and \(V_n\) is compact for all \(n \in \mathbb{N}\). If \(\nu\) is a complex measure on \((X, \mathcal{B}_X)\), then

\[\|\nu\|_{c_0(X)^*} = |\nu|(X).\]
**Proof.** Let $\mu = |\nu|$ and $g : X \to S^1$ be chosen so that $d\nu = gd\mu$. Then for $f \in C_0(X)$,

$$|\nu(f)| = \left| \int_X f d\nu \right| = \left| \int_X fgd\mu \right|$$

$$\leq \int_X |f|d\mu \leq \|f\|_u \cdot \mu(X) = |\nu|(X) \|f\|_u$$

which shows that

$$\|\nu\|_{C_0(X)^*} \leq |\nu|(X).$$

To prove the reverse inequality use Corollary 2.8 to find $f_n \in C_c(X) \subset C_0(X)$ such that $f_n \to \bar{g}$ in $L^1(\mu)$ as $n \to \infty$. Let $g_n = \varphi(f_n)$ where $\varphi : \mathbb{C} \to \mathbb{C}$ is the continuous function in Lemma 2.14 Then

$$|g_n - \bar{g}| \leq |\varphi(f_n) - \bar{g}| \leq |f_n - \bar{g}|$$

and hence $g_n \to \bar{g}$ in $L^1(\mu)$ where now $\|g_n\|_u \leq 1$ and hence

$$\|\nu\|_{C_0(X)^*} \geq |\nu(g_n)| = \left| \int_X g_n gd\mu \right| \to \left| \int_X \bar{g}d\mu \right| = \mu(X) = |\nu|(X).$$

This shows that $\|\nu\|_{C_0(X)^*} \geq |\nu|(X)$ and the proof is complete. \hfill \Box

For completeness, let me now state a form of the Riesz-Markov theorem in the context being considered here.

**Theorem 2.16 (Riesz-Markov Theorem).** Let $(X, \rho)$ be a metric space such that there exists open sets, $\{V_n\}_{n=1}^\infty$, of $X$ such that $V_n \uparrow X$ and $\bar{V}_n$ is compact for all $n \in \mathbb{N}$.

1. If $\varphi$ is a positive linear functional on $C_c(X)$, then there exists a unique positive measure, $\mu$, on $(X, \mathcal{B}_X)$ such that $\mu(K) < \infty$ when $K$ is compact and

   $$\varphi(f) = \mu(f) := \int_X f d\mu$$

   for all $f \in C_c(X)$.

2. If $\varphi \in C_0(X)^*$, then there exists a unique complex measure $\nu$ on $(X, \mathcal{B}_X)$ such that $\varphi(f) = \nu(f)$ for all $f \in C_0(X)$. Moreover the map,

   $$\{\text{complex measures on } X\} \ni \nu \to \left( f \to \nu(f) = \int_X f d\nu \right) \in C_0(X)^*$$

   is an isometric isomorphism of Banach space where $\|\nu\| := |\nu|(X)$ where $\nu$ is a complex measure.
Fourier Series

**Theorem 3.1.** Suppose that $\lambda$ is a complex measure on $(-\pi, \pi), B = B((-\pi, \pi))$. If

$$\int_{(-\pi, \pi)} e^{in\theta} d\nu(\theta) = 0 \text{ for all } n \in \mathbb{Z}$$

then $\nu \equiv 0$.

**Proof.** For $f \in C_c((-\pi, \pi), \mathbb{C})$, let $F(e^{i\theta}) = f(\theta)$ for $-\pi \leq \theta \leq \pi$. Then $F$ is a continuous function on $S^1$ (which is 0 in neighborhood of $-1 \in S^1$) and hence by the Stone-Weierstrass theorem, given $\varepsilon > 0$ there exists $N < \infty$ and $(a_{m,n})_{m,n=0}^N \subset \mathbb{C}$ such that

$$\max_{z \in S^1} \left| F(z) - \sum_{m,n=0}^N a_{m,n} z^m z^n \right| \leq \varepsilon.$$ Evaluating this expression at $z = e^{i\theta}$ then shows

$$\left| f(\theta) - \sum_{m,n=0}^N a_{m,n} e^{i(m-n)\theta} \right| \leq \varepsilon.$$ Therefore

$$\left| \int_{(-\pi, \pi)} f(\theta) d\nu(\theta) \right| = \left| \int_{(-\pi, \pi)} f(\theta) - \sum_{m,n=0}^N a_{m,n} e^{i(m-n)\theta} \right| d\nu(\theta) \leq \varepsilon |\nu|((-\pi, \pi)).$$

As $\varepsilon > 0$ was arbitrary it follows that

$$\int_{(-\pi, \pi)} f(\theta) d\nu(\theta) = 0 \text{ for all } f \in C_c((-\pi, \pi), \mathbb{C})$$

and we have seen this implies $\nu \equiv 0$. $\blacksquare$

**Corollary 3.2.** Let $\mathbb{D} := \text{span}_C \{ \theta \mapsto e^{in\theta} \}_{n \in \mathbb{Z}}$. If $\mu$ is a finite positive measure on $((-\pi, \pi), B = B((-\pi, \pi))$, then $\mathbb{D}$ is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.

**Proof.** First proof. According to the Hahn-Banach theorem, in order to show $\mathbb{D}$ is dense it suffices to show if $\varphi \in L^p(\mu)^*$ satisfies $\varphi|_{\mathbb{D}} \equiv 0$, then $\varphi \equiv 0$. Since $L^p(\mu)^* \cong L^p(\mu)$, there exists $g \in L^p(\mu)$ such that

$$\varphi(f) = \int_{(-\pi, \pi)} f(\theta) g(\theta) d\mu(\theta) \text{ for all } f \in L^p(\mu).$$

Letting $d\nu = gd\mu$ (a complex measure) as $g \in L^p(\mu) \subset L^1(\mu)$, the assumption that $\varphi|_{\mathbb{D}} \equiv 0$ implies

$$0 = \varphi(\theta \mapsto e^{in\theta}) = \int_{(-\pi, \pi)} e^{in\theta} d\nu(\theta) \text{ for all } n \in \mathbb{Z}.$$ From Theorem 3.1, this implies that $\nu \equiv 0$ and hence $d|\nu| = |g| d\mu$ is the zero measure and hence $|g| = 0$ for $\mu$-a.e. Thus $g = 0$ in $L^p(\mu)$ and so $\varphi \equiv 0$.

**Second proof.** In the proof of Theorem 3.1 we have shown every element, $f \in C_c((-\pi, \pi), \mathbb{C})$ may be uniformly approximated by an element of $\mathbb{D}$ and hence in $L^p(\mu)$ for all $1 \leq p < \infty$ because $\mu$ is a finite measure. But we already know that $C_c((-\pi, \pi), \mathbb{C})$ is dense in $L^p(\mu)$ and hence the proof is complete. $\blacksquare$

**Theorem 3.3.** Let $m$ be Lebesgue measure on $(-\pi, \pi)$ and for $f, g \in L^2(m)$, let

$$\langle f | g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \overline{g}(\theta) d\theta.$$ Then $\varphi_n(\theta) := e^{in\theta}$ for $n \in \mathbb{Z}$ forms an orthonormal basis for $L^2(m)$.

The above results easily generalize to the case where $(-\pi, \pi)$ is replaced by $(-\pi, \pi)^d$ for any $d \in \mathbb{N}$. We now setup some more notation.

**Notation 3.4 (Periodic functions)** Let $C_{per}(\mathbb{R}^d)$ denote the $2\pi$-periodic functions in $C(\mathbb{R}^d)$, that is $f \in C_{per}(\mathbb{R}^d)$ iff $f \in C(\mathbb{R}^d)$ and $f(\theta + 2\pi e_i) = f(\theta)$ for all $\theta \in \mathbb{R}^d$ and $i = 1, 2, \ldots, d$. We further let $C^k_{per}(\mathbb{R}^d) = C_{per}(\mathbb{R}^d) \cap C^k(\mathbb{R}^d)$ for all $k \in \mathbb{N}$. Here $\{e_i\}_{i=1}^d$ is the standard basis for $\mathbb{R}^d$.

$^1$ As we have seen, the assertion that $C_c((-\pi, \pi), \mathbb{C})$ is dense in $L^p(\mu)$ holds even if $\mu$ is an infinite measure which is finite on compact sets.
**Definition 3.5.** Let
\[ \mathbb{D} = \text{span}_C \{ \mathbb{R}^d \ni x \mapsto e^{ik \cdot x} \}_{k \in \mathbb{Z}^d} \subset C^\infty_{\text{per}} (\mathbb{R}^d) := \bigcap_k C^k_{\text{per}} (\mathbb{R}^d). \]
In more detail, \( f \in \mathbb{D} \) if and only if there exists a function, \( a : \mathbb{Z}^d \to \mathbb{C} \) with finite support (i.e., \( \# \{ k \in \mathbb{Z}^d : a(k) \neq 0 \} < \infty \) such that
\[ f(x) = f_a(x) = \sum_{k \in \mathbb{Z}^d} a(k) e^{ik \cdot x} \text{ for all } x \in \Omega. \]

**Theorem 3.6 (Density of Trigonometric Polynomials).** Any \( 2\pi \)-periodic continuous function, \( f : \mathbb{R} \to \mathbb{C} \), may be uniformly approximated by a trigonometric polynomial of the form
\[ p(x) = \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda \cdot x} \]
where \( \Lambda \) is a finite subset of \( \mathbb{Z} \) and \( a_\lambda \in \mathbb{C} \) for all \( \lambda \in \Lambda \).

**Proof.** For \( z \in S^1 \), define \( F(z) := f(\theta) \) where \( \theta \in \mathbb{R} \) is chosen so that \( z = e^{i\theta} \). Since \( f \) is \( 2\pi \)-periodic, \( F \) is well defined since if \( \tilde{\theta} \) solves \( e^{i\tilde{\theta}} = z \) then all other solutions are of the form \( \tilde{\theta} + 2\pi n : n \in \mathbb{Z} \). Since the map \( \theta \mapsto e^{i\theta} \) is a local homeomorphism, i.e., for any \( J = (a,b) \) with \( b - a < 2\pi \), the map \( \tilde{\theta} \in J \mapsto \tilde{J} := (e^{i\tilde{\theta}}; \tilde{\theta} \in J) \subset S^1 \) is a homeomorphism, it follows that \( F(z) = f \circ \varphi^{-1}(z) \) for \( z \in \tilde{J} \). This shows \( F \) is continuous when restricted to \( \tilde{J} \). Since such sets cover \( S^1 \), it follows that \( F \) is continuous.

By Example ??, the polynomials in \( z \) and \( \bar{z} = z^{-1} \) are dense in \( C(S^1) \). Hence for any \( \varepsilon > 0 \) there exists
\[ p(z, \bar{z}) = \sum_{0 \leq m, n \leq N} a_{m,n} z^m \bar{z}^n \]
such that \( |F(z) - p(z, \bar{z})| \leq \varepsilon \) for all \( z \in S^1 \). Taking \( z = e^{i\theta} \) then implies
\[ \sup_{\theta} |f(\theta) - p(e^{i\theta}, e^{-i\theta})| \leq \varepsilon \]
where
\[ p(e^{i\theta}, e^{-i\theta}) = \sum_{0 \leq m, n \leq N} a_{m,n} e^{i(m-n)\theta} \]
is the desired trigonometry polynomial.

**Exercise 3.1.** Use Example ?? to show that any \( 2\pi \)-periodic continuous function, \( f : \mathbb{R}^d \to \mathbb{C} \), may be uniformly approximated by a trigonometric polynomial of the form
\[ p(x) = \sum_{\lambda \in \Lambda} a_\lambda e^{i\lambda \cdot x} \]
where \( \Lambda \) is a finite subset of \( \mathbb{Z}^d \) and \( a_\lambda \in \mathbb{C} \) for all \( \lambda \in \Lambda \).

**Hint:** start by showing there exists a unique continuous function, \( F : (S^1)^d \to \mathbb{C} \) such that \( F(e^{i\theta_1}, \ldots, e^{i\theta_d}) = f(x) \) for all \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \).

**Exercise 3.2.** Let \( \Omega = (-\pi, \pi)^d, \mathcal{B} = B_{\Omega} \) be the Borel \( \sigma \)-algebra on \( \Omega, \nu \) be any complex measure on \((\Omega, \mathcal{B})\). Show that \( \nu \equiv 0 \) iff
\[ \int_{\Omega} e^{ik \cdot x} d\nu (x) = 0 \text{ for all } k \in \mathbb{Z}^d. \]

**Hint:** each \( f \in C_c(\Omega, \mathbb{C}) \) may be extended to zero on \( \mathbb{R}^d \setminus \Omega \) and in this way may be viewed as an element of \( C_c(\mathbb{R}^d, \mathbb{C}) \). Using this extended \( f \), let \( F(\theta) := \sum_{k \in \mathbb{Z}^d} f(\theta + 2\pi k) \) so that \( F \in C_{\text{per}}(\mathbb{R}^d) \). Given \( \varepsilon > 0 \), use the Stone–Weierstrass theorem to show there exists \( \Lambda \subset \mathbb{R}^d \) and \( a : \Lambda \to \mathbb{C} \) such that
\[ \sup_{x \in \mathbb{R}^d} \left| F(x) - \sum_{k \in \Lambda} a(k) e^{ik \cdot x} \right| \leq \varepsilon. \] (3.1)

**Exercise 3.3.** Let \( \Omega = (-\pi, \pi)^d, \mathcal{B} = B_{\Omega} \) be the Borel \( \sigma \)-algebra on \( \Omega, \mathcal{B} \) be as in Definition ??, if \( \mu \) is a finite positive measure on \((\Omega, \mathcal{B})\), show \( \mathbb{D} \) is dense in \( L^p(\Omega, \mathcal{B}, \mu) \) for all \( 1 \leq p < \infty \). **Hint:** using \( L^p(\Omega, \mathcal{B}, \mu)^* \cong L^{p'}(\Omega, \mathcal{B}, \mu) \) where \( p' = \frac{p}{p-1} \) and a corollary of the Hahn-Banach theorem, show it suffices to show if \( g \in L^{p'}(\Omega, \mathcal{B}, \mu) \) satisfies,
\[ \int_{\Omega} e^{ik \cdot x} g(x) d\mu (x) = 0 \text{ for all } k \in \mathbb{Z}^d, \] (3.2)
then \( g(x) = 0 \) for \( \mu \)-a.e. \( x \).

### 3.1 Dirichlet Kernel

Although the sum in Eq. (??) is guaranteed to converge relative to the Hilbertian norm on \( H \) it certainly need not converge pointwise even if \( f \in C_{\text{per}}(\mathbb{R}^d) \) as will be proved in Section ?? below. Nevertheless, if \( f \) is sufficiently regular, then the sum in Eq. (??) will converge pointwise as we will now show. In the process we will give a direct and constructive proof of the result in Exercise ??, see Theorem ?? below.

Let us restrict our attention to \( d = 1 \) here. Consider
\[ f_n (\theta) = \sum_{|k| \leq n} \hat{f}(k) \varphi_k (\theta) = \sum_{|k| \leq n} \frac{1}{2\pi} \left[ \int_{[-\pi,\pi]} f(x) e^{-ikx} dx \right] \varphi_k (\theta) \]
\[ = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(x) \sum_{|k| \leq n} e^{ik(\theta-x)} dx \]
\[ = \frac{1}{2\pi} \int_{[-\pi,\pi]} f(x) D_n(\theta-x) dx \]
where
\[ D_n (\theta) := \sum_{k=-n}^{n} e^{ik\theta} \]
is called the Dirichlet kernel. Letting \( \alpha = e^{i\theta/2} \), we have
\[ D_n (\theta) = \sum_{k=-n}^{n} \alpha^{2k} = \frac{\alpha^{2(n+1)} - \alpha^{-2n}}{\alpha^2 - 1} = \frac{\alpha^{2n+1} - \alpha^{-(2n+1)}}{\alpha - \alpha^{-1}} \]
\[ = \frac{2i \sin(n + \frac{1}{2}) \theta}{2i \sin \frac{1}{2} \theta} = \frac{\sin(n + \frac{1}{2}) \theta}{\sin \frac{1}{2} \theta}. \]
and therefore
\[ D_n (\theta) := \sum_{k=-n}^{n} e^{ik\theta} = e^{i(n + \frac{1}{2}) \theta} \sum_{k=-n}^{n} \frac{1}{\sin \frac{1}{2} \theta}, \]
see Figure 3.1, with the understanding that the right side of this equation is 2n + 1 whenever \( \theta \in 2\pi \mathbb{Z} \).

**Figure 3.1.** This is a plot \( D_1 \) and \( D_{10} \).

**Theorem 3.7.** Suppose \( f \in L^1([-\pi,\pi], dm) \) and \( f \) is differentiable at some \( \theta \in [-\pi,\pi] \), then \( \lim_{n \to \infty} f_n (\theta) = f(\theta) \) where \( f_n \) is as in Eq. (3.3).

**Proof.** Observe that
\[ \frac{1}{2\pi} \int_{[-\pi,\pi]} D_n(\theta-x) dx = \frac{1}{2\pi} \int_{[-\pi,\pi]} \sum_{|k| \leq n} e^{ik(\theta-x)} dx = 1 \]
and therefore,
\[ f_n (\theta) - f(\theta) = \frac{1}{2\pi} \int_{[-\pi,\pi]} [f(x) - f(\theta)] D_n(\theta-x) dx \]
\[ = \frac{1}{2\pi} \int_{[-\pi,\pi]} [f(x) - f(\theta-x)] D_n(x) dx \]
\[ = \frac{1}{2\pi} \int_{[-\pi,\pi]} \left[ \frac{f(\theta-x) - f(\theta)}{\sin \frac{1}{2} x} \right] \sin(n + \frac{1}{2}) x dx. \]
If \( f \) is differentiable at \( \theta \), then
\[ \lim_{x \to 0} \frac{f(\theta-x) - f(\theta)}{\sin \frac{1}{2} x} = -2f'(x) \]
and hence there exists \( \varepsilon > 0 \) such that
\[ M_\varepsilon := \sup_{|x| \leq \varepsilon} \left| \frac{f(\theta-x) - f(\theta)}{\sin \frac{1}{2} x} \right| < \infty. \]
Using this remark it is now easily seen that
\[ 1_{[-\pi,\pi]}(x) \frac{f(\theta-x) - f(\theta)}{\sin \frac{1}{2} x} \in L^1([-\pi,\pi], dm) \]
and hence the last expression in Eq. (3.5) tends to 0 as \( n \to \infty \) by the Riemann Lebesgue Lemma, see Corollary ?? or Lemma [1,20].

**Proposition 3.8 (Lack of pointwise convergence).** For each \( \alpha \in [-\pi,\pi]/\sim \), there exists a residual (non-meager set) set \( R_\alpha \subset C_{\text{per}}(\mathbb{R}) \) such that \( \sup_n |f_n (\alpha)| = \infty \) for all \( f \in R_\alpha \)\footnote{Recall this means that \( R_\alpha \) contains a countable union of dense open subsets of \( C_{\text{per}}(\mathbb{R}) \) and such a set is dense in a complete metric space!}. Recall that \( C_{\text{per}}(\mathbb{R}) \) is a complete metric space, hence \( R_\alpha \) is a dense subset of \( C_{\text{per}}(\mathbb{R}) \).

**Proof.** By symmetry considerations, it suffices to assume \( \alpha = 0 \in [-\pi,\pi] \). Let \( dv_n (\theta) := \frac{1}{2\pi} D_n (\theta) d\theta \) which is a complex measure on \((-\pi,\pi)\) which identify with an element of \( C_{\text{per}}(\mathbb{R})^\prime \) by
\[ \nu_n (f) := f_n (0) = \frac{1}{2\pi} \int_{(-\pi, \pi)} f (\theta) D_n (\theta) d\theta. \]

Recall that
\[ \|\nu_n\|_{op} = \frac{1}{2\pi} \|D_n\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n (e^{-i\theta})| d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin (n + \frac{1}{2}) \theta}{\sin \frac{1}{2} \theta} \right| d\theta. \]

Using
\[ |\sin x| = \left| \int_0^x \cos y dy \right| \leq \left| \int_0^x |\cos y| dy \right| \leq |x| \]
in Eq. (3.8) implies that
\[ \|\nu_n\|_{op} \geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin (n + \frac{1}{2}) \theta}{\frac{1}{2} \theta} \right| d\theta = \frac{2}{\pi} \int_0^\pi \left| \frac{\sin (n + \frac{1}{2}) \theta}{\theta} \right| d\theta = \frac{2}{\pi} \int_0^\pi \left| \sin (n + \frac{1}{2}) \theta \right| dy = \int_0^{(n + \frac{1}{2})\pi} |\sin y| dy \rightarrow \infty \text{ as } n \rightarrow \infty \quad (3.6) \]
and hence \( \sup_n \|\nu_n\|_{op} = \infty. \) So by uniform boundedness principal it follows that
\[ R_0 = \{ f \in C_{per} (\mathbb{R}) : \sup_n |\nu_n f| = \infty \} \]
is a residual set. [See Rudin [7, Chapter 5] for more details.]

**Lemma 3.9 (Fourier Series on \( L^1 \)).** For \( f \in L^1((-\pi, \pi)) \), let
\[ \tilde{f} (n) := (f|\varphi_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f (\theta) e^{-in\theta} d\theta \]
Then \( \tilde{f} \in c_0 := C_0 (\mathbb{Z}) \) (i.e. \( \lim_{n \to \infty} \tilde{f} (n) = 0 \)) and the map \( f \in L^1 (T) \xrightarrow{A} \tilde{f} \in c_0 \) is a one to one bounded linear transformation into but not onto \( c_0 \).

**Proof.** By the Riemann Lebesgue Lemma we know that \( \lim_{|n| \to \infty} \tilde{f} (n) = 0 \) so that \( \tilde{f} \in c_0 \) as claimed. Moreover if \( \tilde{f} \equiv 0 \), then by Theorem 3.1 we know that \( d\nu (\theta) := \frac{1}{2\pi} f (\theta) d\theta \) is the zero measure and hence \( f (\theta) = 0 \) for a.e. \( \theta \). This shows that \( A \) is injective. If \( A \) were surjective, the open mapping theorem would imply that \( A^{-1} : c_0 \to L^1 (T) \) is bounded. In particular this implies there exists \( C < \infty \) such that
\[ \|f\|_{L^1} \leq C \|\tilde{f}\|_{c_0} \text{ for all } f \in L^1 (T). \quad (3.7) \]
Taking \( f = D_n \), we find (because \( \tilde{D}_n (k) = 1_{|k| \leq n} \)) that \( \|\tilde{D}_n\|_{c_0} = 1 \) while by Eq. (3.6) \( \lim_{n \to \infty} \|\tilde{D}_n\|_{L^1} = \infty \) contradicting Eq. (3.7). Therefore \( \text{Ran}(A) \neq c_0 \).

### 3.2 Fejér Kernel

Despite the Dirichlet kernel not being positive, it still satisfies the approximate \( \delta \) – sequence property, \( \frac{1}{2\pi} D_n \to \delta_0 \) as \( n \to \infty \), when acting on \( C^1 \) – periodic functions in \( \theta \). In order to improve the convergence properties it is reasonable to try to replace \( \{ f_n : n \in \mathbb{N}_0 \} \) by the sequence of averages (see Exercise ??),
\[ F_N (\theta) = \frac{1}{N + 1} \sum_{n=0}^{N} f_n (\theta) = \frac{1}{N + 1} \sum_{n=0}^{N} \frac{1}{2\pi} \int_{[-\pi, \pi]} f (x) \sum_{|k| \leq n} e^{ik(\theta - x)} dx \]
\[ = \frac{1}{2\pi} \int_{[-\pi, \pi]} K_N (\theta - x) f (x) dx \]
where
\[ K_N (\theta) := \frac{1}{N + 1} \sum_{n=0}^{N} \sum_{|k| \leq n} e^{ik\theta} = \frac{1}{N + 1} \sum_{n=0}^{N} D_n (\theta) \quad (3.8) \]
is the **Fejér kernel** which we now compute more explicitly.

**Lemma 3.10.** The Fejér kernel \( K_N \) in Eq. (3.3) is given by
\[ K_N (\theta) = \frac{1}{N + 1} \frac{\sin^2 \left( \frac{N+1}{2} \theta \right)}{\sin^2 \left( \frac{\theta}{2} \right)} \quad (3.9) \]
and we also have the identity,
\[ K_N (\theta) = \sum_{k=-N}^{N} \left[ 1 - \frac{|k|}{N + 1} \right] e^{ik\theta} \quad (3.10) \]

**Proof.** Recall the trigonometric identities,
\[ \cos (A \pm B) = \cos A \cdot \cos B \mp \sin A \cdot \sin B \]
which subtracted gives
\[ \cos (A + B) - \cos (A - B) = -2 \sin A \cdot \sin B \]
Further taking \( A = B \) in the last equality also gives,
\[ \cos (2A) - 1 = -2 \sin^2 A \]
Using these identities and a telescoping sum argument proves Eq. (3.9) as follows,
\((N + 1) K_N (\theta) := \sum_{n=0}^{N} D_n (\theta)\)
\[
= \frac{1}{2 \sin^2 \frac{\theta}{2} } \sum_{n=0}^{N} 2 \sin \frac{1}{2} \theta \cdot \sin(n + \frac{1}{2}) \theta \\
= -\frac{1}{2 \sin^2 \frac{\theta}{2} } \sum_{n=0}^{N} \left[ \cos ((n + 1) \theta) - \cos (n \theta) \right] \\
= \frac{1}{2 \sin^2 \frac{\theta}{2} } [1 - \cos (N + 1) \theta] \\
= \frac{1}{2 \sin^2 \frac{\theta}{2} } 2 \sin^2 \left( \frac{1}{2} (N + 1) \theta \right).
\]

Equation (3.10) is a consequence of the identity,
\[
(N + 1) K_N (\theta) = \sum_{n=0}^{N} \sum_{|k| \leq n} e^{ik\theta} = \sum_{|k| \leq n \leq N} e^{ik\theta} = \sum_{|k| \leq N} (N + 1 - |k|) e^{ik\theta}.
\]

**Theorem 3.11.** The Fejér kernel \(K_N\) in Eq. (3.8) satisfies:

1. \(K_N (\theta) \geq 0\).
2. \(\frac{1}{2 \pi} \int_{-\pi}^{\pi} K_N (\theta) d\theta = 1\).
3. sup \(\varepsilon \leq |\theta| \leq \pi\) \(K_N (\theta) \to 0\) as \(N \to \infty\) for all \(\varepsilon > 0\), see Figure 3.2.
4. For any continuous \(2\pi\)–periodic function \(f\) on \(\mathbb{R}\), \(K_N \ast f (\theta) \to f (\theta)\) uniformly in \(\theta\) as \(N \to \infty\), where

\[
K_N \ast f (\theta) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} K_N (\theta - \alpha) f (\alpha) d\alpha \\
= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N + 1}\right) \hat{f} (n) e^{in\theta}.
\]

**Proof.** Items 1. is obvious from Eq. (3.9) and item 2. follows from the fact that \(K_N\) is an average of Dirichlet kernels which all integrate to 1, i.e.
\[
\frac{1}{2 \pi} \int_{-\pi}^{\pi} K_N (\theta) d\theta = \frac{1}{N + 1} \sum_{n=0}^{N} \frac{1}{2 \pi} \int_{-\pi}^{\pi} D_n (\theta) d\theta \\
= \frac{1}{N + 1} \sum_{n=0}^{N} 1 = 1.
\]

We can also prove item 2. by integrating Eq. (3.10). Item 3. is a consequence of the elementary estimate;
\[
\sup_{\varepsilon \leq |\theta| \leq \pi} K_N (\theta) \leq \frac{1}{N + 1} \sin^2 \left( \frac{\varepsilon}{2} \right)
\]
and is clearly indicated in Figure 3.2.

Finally, item 4. now follows by the standard approximate \(\delta\)–function arguments, namely,
\[
|K_N \ast f (\theta) - f (\theta)| = \frac{1}{2 \pi} \left| \int_{-\pi}^{\pi} K_N (\theta - \alpha) [f (\alpha) - f (\theta)] d\alpha \right| \\
\leq \frac{1}{2 \pi} \int_{-\pi}^{\pi} K_N (\alpha) |f (\theta - \alpha) - f (\theta)| d\alpha \\
\leq \frac{1}{\pi} \frac{1}{N + 1} \sin^2 \left( \frac{\varepsilon}{2} \right) \frac{1}{\pi} \frac{1}{\sin^4 (\frac{\varepsilon}{2})} \|f\|_{\infty} + \frac{1}{\pi} \int_{|\alpha| \leq \varepsilon} K_N (\alpha) |f (\theta - \alpha) - f (\theta)| d\alpha \\
\leq \frac{1}{\pi} \frac{1}{N + 1} \sin^2 \left( \frac{\varepsilon}{2} \right) \frac{1}{\sin^4 (\frac{\varepsilon}{2})} \|f\|_{\infty} + \frac{1}{\pi} \int_{|\alpha| \leq \varepsilon} |f (\theta - \alpha) - f (\theta)| d\alpha.
\]

Therefore,
\[
\lim_{N \to \infty} \sup_{|\theta| \leq \varepsilon} \|K_N \ast f - f\|_{\infty} \leq \sup_{|\alpha| \leq \varepsilon} \|f (\theta - \alpha) - f (\theta)\| \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.
\]

**3.3 The Dirichlet Problems on \(D\) and the Poisson Kernel**

Let \(D := \{ z \in \mathbb{C} : |z| < 1 \}\) be the open unit disk in \(\mathbb{C} \cong \mathbb{R}^2\), write \(z \in \mathbb{C}\) as \(z = x + iy\) or \(z = re^{i\theta}\), and let \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\) be the **Laplacian** acting on \(C^2 (D)\).
Theorem 3.12 (Dirichlet problem for $D$). To every continuous function $g \in C(\text{bd}(D))$ there exists a unique function $u \in C(D) \cap C^2(D)$ solving

$$\Delta u(z) = 0 \text{ for } z \in D \text{ and } u|_{\partial D} = g.$$ (3.12)

Moreover for $r < 1$, $u$ is given by,

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \alpha) u(e^{i\alpha}) d\alpha =: P_r * u(e^{i\theta})$$ (3.13)

$$= \frac{1}{2\pi} \text{Re} \int_{-\pi}^{\pi} \frac{1 + r e^{i(\theta - \alpha)}}{1 - r e^{i(\theta - \alpha)}} u(e^{i\alpha}) d\alpha$$ (3.14)

where $P_r$ is the Poisson kernel defined by

$$P_r(\delta) := \frac{1}{1 - r^2} \frac{1 - r^2}{1 - 2r \cos \delta + r^2}.$$ (3.12)

(The problem posed in Eq. (3.12) is called the Dirichlet problem for $D$.)

**Proof.** In this proof, we are going to be identifying $S^1 = \text{bd}(D) := \{z \in D : |z| = 1\}$ with $[-\pi, \pi] / (\pi - -\pi)$ by the map $\theta \in [-\pi, \pi] \rightarrow e^{i\theta} \in S^1$.

Also recall that the Laplacian $\Delta$ may be expressed in polar coordinates as,

$$\Delta u = r^{-1} \partial_r \left(r^{-1} \partial_r u\right) + \frac{1}{r^2} \partial_\theta^2 u,$$

where

$$(\partial_r u) (re^{i\theta}) = \frac{\partial}{\partial r} u(re^{i\theta}) \text{ and } (\partial_\theta u) (re^{i\theta}) = \frac{\partial}{\partial \theta} u(re^{i\theta}).$$

**Uniqueness.** Suppose $u$ is a solution to Eq. (3.12) and let

$$\tilde{g}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{ik\theta}) e^{-ik\theta} d\theta$$

and

$$\tilde{u}(r,k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) e^{-ik\theta} d\theta$$ (3.15)

be the Fourier coefficients of $g(\theta)$ and $\theta \rightarrow u(re^{i\theta})$ respectively. Then for $r \in (0,1)$,

$$r^{-1} \partial_r \left(r^{-1} \partial_r \tilde{u}(r,k)\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} r^{-1} \partial_r \left(r^{-1} \partial_r u\right)(re^{i\theta}) e^{-ik\theta} d\theta$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{r^2} \partial_\theta^2 u(re^{i\theta}) e^{-ik\theta} d\theta$$

$$= -\frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) \partial_\theta^2 e^{-ik\theta} d\theta$$

$$= \frac{1}{r^2} k^2 \tilde{u}(r,k)$$

or equivalently

$$r \partial_r (r \partial_r \tilde{u}(r,k)) = k^2 \tilde{u}(r,k).$$ (3.16)

Recall the general solution to

$$r \partial_r (r \partial_r y(r)) = k^2 y(r)$$ (3.17)

may be found by trying solutions of the form $y(r) = r^a$ which then implies $a^2 = k^2$ or $a = \pm k$. From this one sees that $\tilde{u}(r,k)$ solving Eq. (3.16) may be written as $\tilde{u}(r,k) = A_k r^{|k|} + B_k r^{-|k|}$ for some constants $A_k$ and $B_k$ when $k \neq 0$. If $k = 0$, the solution to Eq. (3.17) is gotten by simple integration and the result is $\tilde{u}(r,0) = A_0 + B_0 \ln r$. Since $\tilde{u}(r,k)$ is bounded near the origin for each $k$ it must be that $B_k = 0$ for all $k \in \mathbb{Z}$. Hence we have shown there exists $A_k \in \mathbb{C}$ such that, for all $r \in (0,1)$,

$$A_k r^{|k|} = \tilde{u}(r,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) e^{-ik\theta} d\theta.$$ (3.18)

Since all terms of this equation are continuous for $r \in [0,1]$, Eq. (3.18) remains valid for all $r \in [0,1]$ and in particular we have, at $r = 1$, that

$$A_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{i\theta}) e^{-ik\theta} d\theta = \tilde{g}(k).$$

Hence if $u$ is a solution to Eq. (3.12) then $u$ must be given by

$$u(re^{i\theta}) = \sum_{k \in \mathbb{Z}} \tilde{g}(k) r^{|k|} e^{ik\theta} \text{ for } r < 1.$$ (3.19)

or equivalently,

$$u(z) = \sum_{k \in \mathbb{Z}} \tilde{g}(k) z^k + \sum_{k \in \mathbb{N}_0} \tilde{g}(-k) z^k.$$ (3.19)

Notice that the theory of the Fourier series implies Eq. (3.19) is valid in the $L^2(\partial D)$ - sense. However more is true, since for $r < 1$, the series in Eq. (3.19) is absolutely convergent and in fact defines a $C^\infty$ – function (see Exercise ?? or Corollary ??) which must agree with the continuous function, $\theta \rightarrow u(re^{i\theta})$, for almost every $\theta$ and hence for all $\theta$. This completes the proof of uniqueness.

**Existence.** Given $g \in C(\text{bd}(D))$, let $u$ be defined as in Eq. (3.19). Then, again by Exercise ?? or Corollary ??, $u \in C^\infty(D)$. So to finish the proof it suffices to show $\lim_{r \rightarrow 0^+} u(x) = g(y)$ for all $y \in \text{bd}(D)$. Inserting the formula for $\tilde{g}(k)$ into Eq. (3.19) gives

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \alpha) u(e^{i\alpha}) d\alpha \text{ for all } r < 1$$
where
\[ P_r(\delta) = \sum_{k \in \mathbb{Z}} r^{\lvert k \rvert} e^{i k \delta} = \sum_{k=0}^{\infty} r^k e^{i k \delta} + \sum_{k=0}^{\infty} r^k e^{-i k \delta} - 1 = \]
\[ = \text{Re} \left[ \frac{1 - r e^{i \delta}}{1 - r e^{i \delta}} - 1 \right] = \text{Re} \left[ \frac{1 + r e^{i \delta}}{1 - r e^{i \delta}} \right] \]
\[ = \text{Re} \left[ \frac{(1 + r e^{i \delta}) (1 - r e^{-i \delta})}{|1 - r e^{i \delta}|^2} \right] = \text{Re} \left[ \frac{1 - r^2 + 2ir \sin \delta}{1 - 2r \cos \delta + r^2} \right] \quad (3.20) \]
\[ = \frac{1 - r^2}{1 - 2r \cos \delta + r^2}. \]

The Poisson kernel again solves the usual approximate \( \delta \)-function properties (see Figure 2), namely:

1. \( P_r(\delta) > 0 \) and
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \alpha) \, d\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k \in \mathbb{Z}} r^{\lvert k \rvert} e^{i k (\theta - \alpha)} \, d\alpha
\]
\[ = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} r^{\lvert k \rvert} \int_{-\pi}^{\pi} e^{i k (\theta - \alpha)} \, d\alpha = 1 \]

and

2. \[
\sup_{\varepsilon \leq \lvert \theta \rvert \leq \pi} P_r(\theta) \leq \frac{1 - r^2}{1 - 2r \cos \varepsilon + r^2} \to 0 \text{ as } r \uparrow 1. \]

A plot of \( P_r(\delta) \) for \( r = 0.2, 0.5 \) and 0.7.

Therefore by the same argument used in the proof of Theorem 3.11
\[
\lim_{r \uparrow 1} \sup_{g} \left| u \left( r e^{i \theta} \right) - g \left( e^{i \theta} \right) \right| = \lim_{r \uparrow 1} \sup_{g} \left| (P_r * g) \left( e^{i \theta} \right) - g \left( e^{i \theta} \right) \right| = 0
\]

which certainly implies \( \lim_{x \to y} u(x) = g(y) \) for all \( y \in \text{bd}(D). \)

\[ \Box \]

Remark 3.13 (Harmonic Conjugate). Writing \( z = re^{i \theta} \), Eq. (3.14) may be rewritten as
\[
u(z) = \frac{1}{2\pi} \text{Re} \int_{-\pi}^{\pi} \frac{1 + z e^{-i \alpha}}{1 - z e^{-i \alpha}} u(e^{i \alpha}) \, d\alpha
\]
which shows \( u = \text{Re} F \) where
\[
F(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 + z e^{-i \alpha}}{1 - z e^{-i \alpha}} u(e^{i \alpha}) \, d\alpha.
\]

Moreover it follows from Eq. (3.20) that
\[
\text{Im} F(re^{i \theta}) = \frac{1}{\pi} \text{Im} \int_{-\pi}^{\pi} \frac{r \sin(\theta - \alpha)}{1 - 2r \cos(\theta - \alpha) + r^2} g(e^{i \alpha}) \, d\alpha
\]
\[=: (Q_r * u)(e^{i \theta}). \]

where
\[
Q_r(\delta) := \frac{r \sin(\delta)}{1 - 2r \cos(\delta) + r^2}.
\]

From these remarks it follows that \( v = (Q_r * g)(e^{i \theta}) \) is the harmonic conjugate of \( u \) and \( P_r = Q_r \). For more on this point see Section ?? below.

3.4 Multi-Dimensional Fourier Series

In this subsection we will let \( d\theta, d\,x, d\alpha \), etc. denote standard Lebesgue measure \((m)\) on \( \mathbb{R}^d \), and \( Q := (-\pi, \pi)^d \), and \( H := L^2([-\pi, \pi]^d) \), with inner product given by
\[
\langle f | g \rangle := \left( \frac{1}{2\pi} \right)^d \int_Q f(\theta) \overline{g}(\theta) \, d\theta = \left( \frac{1}{2\pi} \right)^d \int_Q f(\theta) \overline{g}(\theta) \, dm(\theta).
\]

We also let \( \varphi_k(\theta) := e^{i k \cdot \theta} \) for all \( k \in \mathbb{Z}^d \) so that \( \{ \varphi_k \}_{k \in \mathbb{Z}^d} \) is an orthonormal basis for \( H \). For \( f \in L^1(Q) \), we will write \( \hat{f}(k) \) for the Fourier coefficient,
\[
\hat{f}(k) := \langle f | \varphi_k \rangle = \left( \frac{1}{2\pi} \right)^d \int_Q f(\theta) e^{-ik \cdot \theta} \, d\theta. \quad (3.21)
\]

Since any \( 2\pi \)-periodic functions on \( \mathbb{R}^d \) may be identified with function on the \( d \)-dimensional torus, \( T_d \cong \mathbb{R}^d/(2\pi \mathbb{Z})^d \cong (S^1)^d \). I may also write \( C(T^d) \) for \( C_{\text{per}}(\mathbb{R}^d) \) and \( L^p(T^d) \) for \( L^p(Q) \) where elements in \( f \in L^p(Q) \) are to be thought of as there extensions to \( 2\pi \)-periodic functions on \( \mathbb{R}^d \).
Theorem 3.14 (Fourier Series). The functions $\beta := \{\varphi_k : k \in \mathbb{Z}^d\}$ form an orthonormal basis for $H$, i.e. if $f \in H$ then

$$f = \sum_{k \in \mathbb{Z}^d} \langle f | \varphi_k \rangle \varphi_k = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \varphi_k$$

(3.22)

where the convergence takes place in $L^2([-\pi, \pi]^d)$.

Proof. Simple computations show $\beta := \{\varphi_k : k \in \mathbb{Z}^d\}$ is an orthonormal set. This fact coupled with Exercise ?? which states span $\beta$ is dense in $L^2([\pi, \pi]^d)^3$ completes the proof. 

3.5 Translation Invariant Operators

Proposition 3.15. Consider, for $f \in L^2([\pi, \pi])$ which we identify with $2\pi$-periodic functions. Let $f_\alpha(\theta) := f(\theta - \alpha) = U_\alpha f$ for $\alpha \in \mathbb{R}$ which is now unitary operator on $L^2$. Suppose that $T \subseteq B(L^2([\pi, \pi]))$ and $T_\alpha = U_\alpha T$ for all $\alpha \in \mathbb{R}$, then $T \varphi_n = \lambda_n \varphi_n$ for some $\lambda_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$. If we further assume that $\sum_n |\lambda_n| < \infty$, then

$$(Tf)(\theta) = \int_{-\pi}^{\pi} k(\theta - \alpha) f(\alpha) d\alpha \text{ for a.e. } \theta,$$

(3.23)

where

$$k(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \lambda_n e^{in\theta}.$$

Proof. Since $U_\alpha \varphi_n = e^{-i\alpha n} \varphi_n$ we have $e^{-i\alpha n} T \varphi_n = T U_\alpha \varphi_n = U_\alpha T \varphi_n$ and then taking the inner product of this equation with $\varphi_m$ shows

$$e^{-i\alpha n} \langle T \varphi_n | \varphi_m \rangle = \langle U_\alpha T \varphi_n | \varphi_m \rangle = \langle T \varphi_n | U_\alpha \varphi_m \rangle = \langle T \varphi_n | e^{-i\alpha m} \varphi_m \rangle = e^{-i\alpha n} \langle T \varphi_n | \varphi_m \rangle$$

for all $\alpha \in \mathbb{R}$.

From this it follows that

$$\langle T \varphi_n | \varphi_m \rangle = 0 \text{ if } n \neq m$$

and hence

$$T \varphi_n = \sum_{m \in \mathbb{Z}} \langle T \varphi_n | \varphi_m \rangle \varphi_m = \langle T \varphi_n | \varphi_n \rangle \varphi_n = \lambda_n \varphi_n$$

where $\lambda_n := \langle T \varphi_n | \varphi_n \rangle$.

Now let us further suppose that $\sum_{n \in \mathbb{Z}} |\lambda_n| < \infty$. Then for $f \in L^2([\pi, \pi])$ we have

$$Tf = T \sum_{n \in \mathbb{Z}} \langle f | \varphi_n \rangle \varphi_n = \sum_{n \in \mathbb{Z}} \langle f | \varphi_n \rangle T \varphi_n = \sum_{n \in \mathbb{Z}} \lambda_n \langle f | \varphi_n \rangle \varphi_n.$$ 

Now (leaving the details to the reader) we have

$$\sum_{n \in \mathbb{Z}} \lambda_n \langle f | \varphi_n \rangle \varphi_n(\theta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \lambda_n \int_{-\pi}^{\pi} \alpha f(\alpha) e^{-i\alpha \theta} \, d\alpha$$

$$= \int_{-\pi}^{\pi} \alpha f(\alpha) \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \lambda_n e^{-i\alpha \theta} \, d\alpha$$

where the sums are pointwise convergent uniformly in $\theta$. The usual arguments now shows that

$$\langle Tf | \varphi_n \rangle = \int_{-\pi}^{\pi} k(\theta - \alpha) f(\alpha) d\alpha \text{ for a.e. } \theta.$$

Exercise 3.4. Suppose that $(X, B, \mu)$ is a $\sigma$-finite measure space and $B : L^2(\mu) \rightarrow L^2(\mu)$ is a bounded linear operator such that $[B, M_\psi] = 0$ for all $\varphi : X \rightarrow \mathbb{C}$ which are bounded and measurable. Show there exists a bounded measurable function, $\psi : X \rightarrow \mathbb{C}$ such that $B = M_\psi$.

Exercise 3.5. Let $\mu$ be a $\sigma$-finite measure on $(\mathbb{R}^d, B)$ and suppose that $B : L^2(\mu) \rightarrow L^2(\mu)$ is a bounded linear operator such that $[B, M_\varphi] = 0$ for all $\lambda \in \mathbb{R}^d$ where $\varphi_\lambda(x) = e^{i\lambda \cdot x}$ for all $\lambda \in \mathbb{R}^d$. Show $B = M_\psi$ for some $\psi \in L^\infty(\mu)$.

Corollary 3.16. If $B : L^2(m) \rightarrow L^2(m)$ is an operator such that $BT_a = T_a B$ for all $a \in \mathbb{R}^d$ where $(T_a f)(\cdot) = f(\cdot - a)$ for each $a \in \mathbb{R}^d$, then $-B^* F = M_\psi$ for some $\psi \in L^\infty(m)$. The converse holds as well.

Proof. This is just a matter of noting that the given assumptions on $B$ holds iff $[F, B^* M_\varphi] = 0$ for all $a \in \mathbb{R}^d$. We then apply the previous results.
Convolutions on $\mathbb{R}^d$

Throughout this chapter we will be solely concerned with $d$-dimensional Lebesgue measure, $m$, and we will simply write $L^p$ for $L^p(\mathbb{R}^d, m)$. The main object of study here is the convolution of two functions.

**Definition 4.1 (Convolution).** Let $f, g : \mathbb{R}^d \to \mathbb{C}$ be measurable functions. We define

$$f \ast g (x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy$$

whenever the integral is defined, i.e. either $f \in L^1(\mathbb{R}^d, m)$ or $f \ast g \geq 0$. Notice that the condition that $f \ast g \in L^1(\mathbb{R}^d, m)$ is equivalent to writing $|f| \ast |g| (x) < \infty$. By convention, if the integral in Eq. (1.1) is not defined, let $f \ast g(x) := 0$.

**Notation 4.2** Given a multi-index $\alpha \in \mathbb{Z}^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$,

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \text{ and } \partial_x^\alpha := \prod_{j=1}^d \left( \frac{\partial}{\partial x_j} \right)^{\alpha_j}.$$

For $z \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{C}$, let $\tau_z f : \mathbb{R}^d \to \mathbb{C}$ be defined by

$$\tau_z f(x) = f(x-z).$$

**Remark 4.3 (The Significance of Convolution).**

1. Suppose that $f, g \in L^1(m)$ are positive functions and let $\mu$ be the measure on $(\mathbb{R}^d)^2$ defined by

$$d\mu(x, y) := f(x) g(y) dm(x) dm(y).$$

Then if $h : \mathbb{R} \to [0, \infty]$ is a measurable function we have

$$\int_{(\mathbb{R}^d)^2} h(x+y) d\mu(x, y) = \int_{(\mathbb{R}^d)^2} h(x+y) f(x) g(y) dm(x) dm(y)$$

$$= \int_{(\mathbb{R}^d)^2} h(x) f(x-y) g(y) dm(x) dm(y)$$

$$= \int_{\mathbb{R}^d} h(x) f \ast g(x) dm(x).$$

In other words, this shows the measure $(f \ast g)_m$ is the same as $S \mu$ where $S(x,y) := x+y$. In probability lingo, the distribution of a sum of two “independent” (i.e. product measure) measurable functions is the the convolution of the individual distributions.

2. Suppose that $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$ is a constant coefficient differential operator and suppose that we can solve (uniquely) the equation $Lu = g$ in the form

$$u(x) = Kg(x) := \int_{\mathbb{R}^d} k(x, y) g(y) dy$$

where $k(x, y)$ is an “integral kernel.” (This is a natural sort of assumption since, in view of the fundamental theorem of calculus, integration is the inverse operation to differentiation.) Since $\tau_z L = L \tau_z$ for all $z \in \mathbb{R}^d$, (this is another way to characterize constant coefficient differential operators) and $L^{-1} = K$ we should have $\tau_z K = L \tau_z$. Writing out this equation then says

$$\int_{\mathbb{R}^d} k(x-z, y) g(y) dy = (Kg)(x-z) = \tau_z Kg(x) = (K \tau_z g)(x)$$

$$= \int_{\mathbb{R}^d} k(x, y) g(y-z) dy = \int_{\mathbb{R}^d} k(x, y+z) g(y) dy.$$

Since $g$ is arbitrary we conclude that $k(x-z, y) = k(x, y+z)$. Taking $y = 0$ then gives

$$k(x, z) = k(x-z, 0) := \rho(x-z).$$

We thus find that $Kg = \rho \ast g$. Hence we expect the convolution operation to appear naturally when solving constant coefficient partial differential equations. More about this point later.

### 4.1 Young’s Inequalities

**Proposition 4.4.** Suppose $p \in [1, \infty]$, $f \in L^1$ and $g \in L^p$, then $f \ast g(x)$ exists for almost every $x$, $f \ast g \in L^p$ and

$$\|f \ast g\|_p \leq \|f\|_1 \|g\|_p.$$

**Proof.** This follows directly from Minkowski’s inequality for integrals. ■
Proposition 4.5. Suppose that $p \in [1, \infty)$, then $\tau_z : L^p \to L^p$ is an isometric isomorphism and for $f \in L^p$, $z \in \mathbb{R}^d \to \tau_z f \in L^p$ is continuous.

Proof. The assertion that $\tau_z : L^p \to L^p$ is an isometric isomorphism follows from translation invariance of Lebesgue measure and the fact that $\tau_z \circ \tau_z = \text{id}$. When $g \in C_c (\mathbb{R}^d)$ a relatively simple use of the dominated convergence theorem shows $\lim_{z \to z_0} \| \tau_z g - \tau_{z_0} g \|_p = 0$, i.e. $z \in \mathbb{R}^d \to \tau_z g \in L^p$ is continuous.

As $C_c (\mathbb{R}^d)$ is dense in $L^p (\mathbb{R}^d)$, for any $f \in L^p$ there exists $f_n \in C_c (\mathbb{R}^d)$ such that $\lim_{n \to \infty} \| f - f_n \|_p = 0$. It then follows that

$$\sup_{z \in \mathbb{R}^d} \| \tau_z f - \tau_z f_n \|_p = \| f - f_n \|_p \to 0 \text{ as } n \to \infty$$

and hence $\mathbb{R}^d \ni z \to \tau_z f \in L^p (m)$ is the uniform limit of continuous functions, $z \to \tau_z f_n$, and therefore is itself continuous.

Definition 4.6. Suppose that $(X, \tau)$ is a topological space and $\mu$ is a measure on $\mathcal{B}_X = \sigma(\tau)$. For a measurable function $f : X \to \mathbb{C}$ we define the essential support of $f$ by

$$\text{supp}_\mu (f) = \{ x \in X : \mu(\{ y \in V : f(y) \neq 0 \}) > 0 \, \forall \text{ neighborhoods } V \text{ of } x \}. \tag{4.2}$$

Equivalently, $x \notin \text{supp}_\mu (f)$ iff there exists an open neighborhood $V$ of $x$ such that $1_V f = 0$ a.e.

It is not hard to show that if $\text{supp}(\mu) = X$ (see Definition ??) and $f \in C (X)$ then $\text{supp}_\mu (f) = \text{sup} (f := \{ f \neq 0 \}, \text{ see Exercise ??}.

Lemma 4.7. Suppose $(X, \tau)$ is second countable and $f : X \to \mathbb{C}$ is a measurable function and $\mu$ is a measure on $\mathcal{B}_X$. Then $X := \mathcal{U} \cup \text{supp}_\mu (f)$ may be described as the largest open set $W$ such that $f \in W (x) = 0$ for $\mu$-a.e. $x$. Equivalently, put

$C := \text{supp}_\mu (f)$ is the smallest closed subset of $X$ such that $f = f \chi_C$ a.e.

Proof. To verify that the two descriptions of $\text{supp}_\mu (f)$ are equivalent, suppose $\text{supp}_\mu (f)$ is defined as in Eq. (4.2) and $W := X \setminus \text{supp}_\mu (f)$. Then

$$W = \{ x \in X : \exists \, \tau \ni \mathcal{U} \ni \exists \, \tau \text{ such that } \mu(\{ y \in V : f(y) \neq 0 \}) = 0 \}
= \cup \{ V \subset \mathcal{U} : \mu(\{ f \neq 0 \}) = 0 \}
= \{ V \subset \mathcal{U} : f \chi_V = 0 \text{ for } \mu\text{-a.e.} \}.$$

So to finish the argument it suffices to show $\mu(\{ f \neq 0 \}) = 0$. To do this let $\mathcal{U}$ be a countable base for $\tau$ and set

$$\mathcal{U}_f := \{ V \subset \mathcal{U} : f \chi_V = 0 \text{ a.e.} \}.$$
Remark 4.9. Let $A, B$ be closed sets of $\mathbb{R}^d$, it is not necessarily true that $A + B$ is still closed. For example, take

$$A = \{(x, y) : x > 0 \text{ and } y \geq 1/x\} \quad \text{and} \quad B = \{(x, y) : x < 0 \text{ and } y \geq 1/|x|\},$$

then every point of $A + B$ has a positive $y$-component and hence is not zero. On the other hand, for $x > 0$ we have $(x, 1/x) + (-x, 1/x) = (0, 2/x) \in A + B$ for all $x$ and hence $0 \in A + B$ showing $A + B$ is not closed. Nevertheless if one of the sets $A$ or $B$ is compact, then $A + B$ is closed again. Indeed, if $A$ is compact and $x_n = a_n + b_n \in A + B$ and $x_n \to x \in \mathbb{R}^d$, then by passing to a subsequence if necessary we may assume $\lim_{n \to \infty} a_n = a = A$ exists. In this case

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} (x_n - a_n) = x - a \in B$$

exists as well, showing $x = a + b \in A + B$.

Proposition 4.10. Suppose that $p, q \in [1, \infty]$ and $p$ and $q$ are conjugate exponents, $f \in L^p$ and $g \in L^q$, then $f * g \in \text{BC}(\mathbb{R}^d)$ with $f * g$ being uniformly continuous and satisfying, $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. If we further assume that $p, q \in (1, \infty) \text{ then } f * g \in C_0(\mathbb{R}^d)$.

Proof. The existence of $f * g(x)$ and the estimate $|f * g(x)| \leq \|f\|_p \|g\|_q$ for all $x \in \mathbb{R}^d$ is a simple consequence of Hölder’s inequality and the translation invariance of Lebesgue measure. In particular this shows $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$. By relabeling $p$ and $q$ if necessary we may assume that $p \in [1, \infty)$. Since

$$\|\tau_{z}(f * g) - f * g\|_\infty = \|\tau_{z}(f * g) - f * g\|_u \leq \|\tau_{z}(f - f)\|_p \|g\|_q \to 0 \text{ as } z \to 0$$

it follows that $f * g$ is uniformly continuous. Finally if $p, q \in (1, \infty)$, we learn from Lemma 4.8 and what we have just proved that $f_m * g_m \in C_c(\mathbb{R}^d)$ where $f_m = f 1_{|f| \leq m}$ and $g_m = g 1_{|g| \leq m}$. Moreover,

$$\|f * g - f_m * g_m\|_\infty \leq \|f - f_m\|_p \|g\|_q + \|f_m * g - f_m * g_m\|_\infty \leq \|f - f_m\|_p \|g\|_q + \|f_m\|_p \|g - g_m\|_q \to 0 \text{ as } m \to \infty$$

showing $f * g \in C_0(\mathbb{R}^d)$ as $C_0(\mathbb{R}^d)$ is closed under uniform convergence. □

Theorem 4.11 (Young’s Inequality). Let $p, q, r \in [1, \infty]$ satisfy

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

(4.3)

If $f \in L^p$ and $g \in L^q$ then $|f| * |g|(x) < \infty \text{ for m-a.e. } x$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

(4.4)

In particular $L^1$ is closed under convolution. (The space $(L^1, \ast)$ is an example of a “Banach algebra” without unit.) [See Section 46 for an interpolation proof of this theorem.]

Proof. By the usual sorts of arguments, we may assume $f$ and $g$ are positive functions. Let $\alpha, \beta \in [0, 1]$ and $p_1, p_2 \in [0, \infty]$ satisfy $p_1^{-1} + p_2^{-1} + r^{-1} = 1$. Then by Hölder’s inequality,

$$f * g(x) = \int_{\mathbb{R}^d} \left[ f(x - y) \frac{1 - \alpha}{\alpha} g(y) \frac{1 - \beta}{\beta} \right] f(x - y) \frac{\alpha}{\alpha} g(y) \frac{\beta}{\beta} dy \leq \left( \int_{\mathbb{R}^d} f(x - y) \frac{1 - \alpha}{\alpha} g(y) \frac{1 - \beta}{\beta} dy \right)^{1/r} \left( \int_{\mathbb{R}^d} f(x - y) \frac{\alpha}{\alpha} g(y) \frac{\beta}{\beta} dy \right)^{1/p_1} \times \left( \int_{\mathbb{R}^d} g(y) \frac{\alpha}{\alpha} g(y) \frac{\beta}{\beta} dy \right)^{1/p_2} \leq \left( \int_{\mathbb{R}^d} f(x - y) \frac{1 - \alpha}{\alpha} g(y) \frac{1 - \beta}{\beta} dy \right)^{1/r} \|f\|_{\alpha p_1} \|g\|_{\beta p_2} \cdot$$

Taking the $r$th power of this equation and integrating on $x$ gives

$$\|f * g\|_r \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f(x - y) \frac{1 - \alpha}{\alpha} g(y) \frac{1 - \beta}{\beta} dy \right) dx \cdot \|f\|_{\alpha p_1} \|g\|_{\beta p_2} \cdot$$

(4.5)

Let us now suppose, $(1 - \alpha)r = \alpha p_1$ and $(1 - \beta)r = \beta p_2$, in which case Eq. (4.5) becomes,

$$\|f * g\|_r \leq \|f\|_{\alpha p_1} \|g\|_{\beta p_2},$$

which is Eq. (4.4) with

$$p := (1 - \alpha)r = \alpha p_1 \text{ and } q := (1 - \beta)r = \beta p_2.$$

So to finish the proof, it suffices to show $p$ and $q$ are arbitrary indices in $[1, \infty]$ satisfying $p^{-1} + q^{-1} = 1 + r^{-1}$. If $\alpha, \beta, p_1, p_2$ satisfy the relations above, then

$$\alpha = \frac{r}{p + p_1} \quad \text{and} \quad \beta = \frac{r}{p + p_2}$$

and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha p_1} + \frac{1}{\alpha p_2} = \frac{r + p_1}{p_1 r} + \frac{r + p_2}{p_2 r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{2}{r} = 1 + \frac{1}{r}.$$
Conversely, if $p, q, r$ satisfy Eq. (4.3), then let $\alpha$ and $\beta$ satisfy $p = (1 - \alpha)r$ and $q = (1 - \beta)r$, i.e.

$$\alpha := \frac{r - p}{r} = 1 - \frac{p}{r} \leq 1 \quad \text{and} \quad \beta := \frac{r - q}{r} = 1 - \frac{q}{r} \leq 1.$$  

Using Eq. (4.3) we may also express $\alpha$ and $\beta$ as

$$\alpha = p(1 - \frac{1}{q}) \geq 0 \quad \text{and} \quad \beta = q(1 - \frac{1}{p}) \geq 0$$

and in particular we have shown $\alpha, \beta \in [0, 1]$. If we now define $p_1 := p/\alpha \in (0, \infty]$ and $p_2 := q/\beta \in (0, \infty]$, then

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{r} = (1 - \frac{1}{q}) + (1 - \frac{1}{p}) + \frac{1}{r} = 2 - \left(1 + \frac{1}{\alpha}\right) + \frac{1}{r} = 1$$

as desired. $\blacksquare$

**Remark 4.12.** Here is a scaling argument that explains why Eq. (4.3) is the only possible relationship for which Eq. (4.4) can hold. For $\lambda > 0$, let $f_\lambda(x) := f(\lambda x)$, then after a few simple change of variables we find

$$\|f_\lambda\|_p = \lambda^{-d/p} \|f\|_p \quad \text{and} \quad (f \ast g)_\lambda = \lambda^d f_\lambda \ast g_\lambda.$$  

Therefore if Eq. (4.4) holds for some $p,q,r \in [1, \infty]$, we would also have

$$\|f \ast g\|_r = \lambda^{d/r} \|(f \ast g)_\lambda\|_r \leq \lambda^{d/r} \|f_\lambda\|_p \|g_\lambda\|_q = \lambda^{(d+d/r-d/p-d/q)} \|f\|_p \|g\|_q$$

for all $\lambda > 0$. This is only possible if Eq. (4.3) holds.

**Theorem 4.13 (Approximate $\delta$-functions).** Let $p \in [1, \infty]$, $\varphi \in L^1(\mathbb{R}^d)$, $a := \int_{\mathbb{R}^d} \varphi(x) \, dx$, and for $t > 0$ let $\varphi_t(x) = t^{-d} \varphi(x/t)$. Then

1. If $f \in L^p$ with $p < \infty$ then $\varphi_t \ast f \to af$ in $L^p$ as $t \downarrow 0$.
2. If $f \in BC(\mathbb{R}^d)$ and $f$ is uniformly continuous then $||\varphi_t \ast f - af||_\infty \to 0$ as $t \downarrow 0$.
3. If $f \in L^\infty$ and $f$ is continuous on $U \subset \mathbb{R}^d$ then $\varphi_t \ast f \to af$ uniformly on compact subsets of $U$ as $t \downarrow 0$.

(See Proposition 4.27 below and for a statement about almost everywhere convergence.)

**Proof.** Making the change of variables $y = tz$ implies

$$\varphi_t \ast f(x) = \int_{\mathbb{R}^d} f(x - y) \varphi_t(y) \, dy = \int_{\mathbb{R}^d} f(x - tz) \varphi(y) \, dz$$

so that

$$\varphi_t \ast f(x) - af(x) = \int_{\mathbb{R}^d} [f(x - tz) - f(x)] \varphi(y) \, dz \leq \int_{\mathbb{R}^d} |f(x - tz) - f(x)| \varphi(y) \, dz.$$  

Hence by Minkowski’s inequality for integrals, Proposition 4.5, and the dominated convergence theorem,

$$||\varphi_t \ast f - af||_p \leq \int_{\mathbb{R}^d} |||\varphi_t \ast f - f||_p \varphi(z)|| \, dz \to 0 \text{ as } t \downarrow 0.$$  

Item 2. is proved similarly. Indeed, form Eq. (4.7)

$$||\varphi_t \ast f - af||_\infty \leq \int_{\mathbb{R}^d} |||\varphi_t \ast f - f||_\infty \varphi(z)|| \, dz$$

which again tends to zero by the dominated convergence theorem because $\lim_{t \downarrow 0} ||\varphi_t \ast f - f||_\infty = 0$ uniformly in $z$ by the uniform continuity of $f$.

Item 3. Let $B_R := B(0; R)$ be a large ball in $\mathbb{R}^d$ and $K \subset U$, then

$$\sup_{x \in K} |\varphi_t \ast f(x) - af(x)|$$

$$\leq \int_{B_R} |f(x - tz) - f(x)| \varphi(z) \, dz + \int_{B_R} |f(x - tz) - f(x)| \varphi(z) \, dz$$

$$\leq \int_{B_R} |\varphi(z)| \, dz \cdot \sup_{x \in K, z \in B_R} |f(x - tz) - f(x)| + 2 \|f\|_\infty \int_{B_R} |\varphi(z)| \, dz$$

$$\leq \|\varphi\|_1 \cdot \sup_{x \in K, z \in B_R} |f(x - tz) - f(x)| + 2 \|f\|_\infty \int_{\mathbb{R}^d} |\varphi(z)| \, dz$$

so that using the uniform continuity of $f$ on compact subsets of $U$,

$$\limsup_{t \downarrow 0} \sup_{x \in K} |\varphi_t \ast f(x) - af(x)| \leq 2 \|f\|_\infty \int_{|z| > R} |\varphi(z)| \, dz \to 0 \text{ as } R \to \infty.$$  

We will often wish to take $\varphi$ in Theorem 4.13 to be a smooth function with compact support. The existence of such functions is a simple consequence of the result of the next exercise, see Lemma 4.14.
Exercise 4.1. Let

\[ f(t) = \begin{cases} 
     e^{-1/t} & \text{if } t > 0 \\
     0 & \text{if } t \leq 0.
\end{cases} \]

Show \( f \in C^\infty(\mathbb{R}, [0, 1]) \). Hints: you might start by first showing \( \lim_{t \to 0} f(t) = 0 \) for all \( n \in \mathbb{N} \).

Lemma 4.14 (Smooth bump functions). There exists \( \phi \in C_c^\infty(\mathbb{R}^d, [0, 0]) \) such that \( \phi(0) > 0 \), \( \supp(\phi) \subset B(0, 1) \) and \( \int_{\mathbb{R}^d} \phi(x) \, dx = 1 \).

Proof. Define \( h(t) = f(t)(1-t)f(t+1) \) where \( f \) is as in Exercise 4.1. Then \( h \in C_c^\infty(\mathbb{R}, [0, 1]) \), \( \supp(h) \subset [-1, 1] \) and \( h(0) = e^{-2} > 0 \). Define \( c = \int_{\mathbb{R}^d} h(|x|^2) \, dx \).

Then \( \phi(x) = c^{-1} h(|x|^2) \) is the desired function.

The reader asked to prove the following proposition in Exercise ?? below.

Proposition 4.15. Suppose that \( f \in L^1_{loc}(\mathbb{R}^d, m) \) and \( \phi \in C^1_c(\mathbb{R}^d) \), then \( \int f * \phi \) is continuous and \( \partial_i (f * \phi) = f * \partial_i \phi \). Moreover if \( \phi \in C_c^\infty(\mathbb{R}^d) \) then \( f * \phi \in C_c^\infty(\mathbb{R}^d) \).

The existence of smooth bump functions along with Proposition 4.15 allows us to construct smooth functions approximating most any function we like. Here are some useful results along this vein.

Corollary 4.16. Let \( X \subset \mathbb{R}^d \) be an open set and \( \mu \) be a \( K \)-finite measure on \( \mathcal{B}_X \).

1. Then \( C^\infty_c(X) \) is dense in \( L^p(\mu) \) for all \( 1 \leq p < \infty \).
2. If \( h \in L^1_{loc}(\mu) \) satisfies

\[ \int_X f h \, d\mu = 0 \text{ for all } f \in C^\infty_c(X) \]  \hspace{1cm} (4.8)

then \( h(x) = 0 \) for \( \mu \)-a.e. \( x \).

Proof. Let \( f \in C^\infty_c(X) \), \( \phi \) be as in Lemma 4.14 \( \phi \) be as in Theorem 4.13 and set \( \psi_t := \phi \ast (f 1_X) \). Then by Proposition 4.15 \( \psi_t \in C^\infty_c(X) \) and by Lemma 4.18 there exists a compact set \( K \subset X \) such that \( \supp(\psi_t) \subset K \) for all \( t \) sufficiently small. By Theorem 4.13 \( \psi_t \to f \) uniformly on \( X \) as \( t \downarrow 0 \).

1. The dominated convergence theorem (with dominating function being \( \|f\|_\infty 1_K \)), shows \( \psi_t \to f \) in \( L^p(\mu) \) as \( t \downarrow 0 \). This proves Item 1., since Theorem ?? guarantees that \( C_c^\infty(X) \) is dense in \( L^p(\mu) \).

2. Keeping the same notation as above, the dominated convergence theorem (with dominating function being \( \|f\|_\infty |h| 1_{\mathbb{R}^d} \)) implies

\[ 0 = \lim_{t \to 0} \int_X \psi_t h d\mu = \int_X \lim_{t \to 0} \psi_t h d\mu = \int_X f h d\mu. \]

The proof is now finished by an application of Lemma ??.

Lemma 4.17. Given a rectangle \( R \) in \( \mathbb{R}^d \), say \( R = [a_1, b_1] \times \cdots \times [a_n, b_n] \), then there exists \( f_k \in C_c^\infty(\mathbb{R}^d) \) such that \( f_k \to 1_R \) boundedly.

Proof. It suffices to consider the one dimensional case. Let \( \varphi \in C_c^\infty(\mathbb{R}) \) such that \( \varphi \geq 0 \), \( \varphi \) is supported in \((-1, 0)\) and \( \int_{\mathbb{R}} \varphi(x) \, dx = 1 \). Set \( \varphi_c(x) = \frac{1}{c} \varphi(\frac{x}{c}) \).

Then

\[ \varphi_c \ast 1_{[a,b]}(x) = \int_{\mathbb{R}} \varphi_c(y) 1_{[a,b]}(x-y) \, dy = \int_{\mathbb{R}} \varphi(y) 1_{[a,b]}(x-\varepsilon y) \, dy \]

\[ = \int_{-\varepsilon}^{\varepsilon} \varphi(y) 1_{[a,b]}(x-\varepsilon y) \, dy \to 1_{[a,b]}(x) \text{ as } \varepsilon \downarrow 0 \]

for all \( x \in \mathbb{R} \).

Corollary 4.18 (\( C^\infty \) – Uryson’s Lemma). Given \( K \subset \mathbb{R}^d \), there exists \( f \in C_c^\infty(\mathbb{R}^d, [0, 1]) \) such that \( \supp(f) \subseteq U \) and \( f = 1_K \).

Proof. Let \( d \) be the standard metric on \( \mathbb{R}^d \) and \( \varepsilon := d(K, U^c) \) which is positive since \( K \) is compact and \( d(x, U^c) > 0 \) for all \( x \in K \). Further let \( V := \{ x \in \mathbb{R}^d : d(x, K) < \varepsilon/3 \} \) and then take \( f = \varphi_{\varepsilon/3} \ast 1_V \) where \( \varphi_t(x) = t^{-d} \varphi(x/t) \) as in Theorem 4.13 and \( \varphi \) is as in Lemma 4.14. It then follows that

\[ \supp(f) \subseteq \supp(\varphi_{\varepsilon/3}) + V_{\varepsilon/3} \subseteq V_{2\varepsilon/3} \subseteq U. \]

Since \( V_{2\varepsilon/3} \) is closed and bounded, \( f \in C_c^\infty(U) \) and for \( x \in K \),

\[ f(x) = \int_{\mathbb{R}^d} 1_{d(y,K)<\varepsilon/3} \cdot \varphi_{\varepsilon/3}(x-y) \, dy = \int_{\mathbb{R}^d} \varphi_{\varepsilon/3}(x-y) \, dy = 1. \]

The proof will be finished after the reader (easily) verifies \( 0 \leq f \leq 1 \).

Here is an application of this Corollary 4.18 whose proof is left to the reader, Exercise ??.

Lemma 4.19 (Integration by Parts). Suppose \( f \) and \( g \) are measurable functions on \( \mathbb{R}^d \) such that \( t \to f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d) \) and \( t \to g(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_d) \) are continuously differentiable functions on \( \mathbb{R}^d \) for each fixed \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \). Moreover assume \( f \cdot g, \frac{\partial f}{\partial x_i} \cdot g \) and \( f \cdot \frac{\partial g}{\partial x_i} \) are in \( L^1(\mathbb{R}^d, m) \). Then

\[ \int_{\mathbb{R}^d} \frac{\partial f}{\partial x_i} \cdot g \, dm = - \int_{\mathbb{R}^d} f \cdot \frac{\partial g}{\partial x_i} \, dm. \]

With this result we may give another proof of the Riemann Lebesgue Lemma.
Lemma 4.20 (Riemann Lebesgue Lemma). For \( f \in L^1(\mathbb{R}^d, m) \) let
\[
\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dm(x)
\]
be the Fourier transform of \( f \). Then \( \hat{f} \in C_0(\mathbb{R}^d) \) and \( \|\hat{f}\|_{L^\infty} \leq (2\pi)^{-d/2} \|f\|_1 \).
(The choice of the normalization factor, \((2\pi)^{-d/2}\), in \( \hat{f} \) is for later convenience.)

**Proof.** The fact that \( \hat{f} \) is continuous is a simple application of the dominated convergence theorem. Moreover,
\[
|\hat{f}(\xi)| \leq \int_{\mathbb{R}^d} |f(x)| \, dm(x) \leq (2\pi)^{-d/2} \|f\|_1
\]
so it only remains to see that \( \hat{f}(\xi) \to 0 \) as \( |\xi| \to \infty \). First suppose that \( f \in C^\infty_c(\mathbb{R}^d) \) and let \( \Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} \) be the Laplacian on \( \mathbb{R}^d \). Notice that
\[
\frac{\partial}{\partial x_j} e^{-i\xi \cdot x} = -i\xi_j e^{-i\xi \cdot x} \quad \text{and} \quad \Delta e^{-i\xi \cdot x} = -|\xi|^2 e^{-i\xi \cdot x}.
\]
Using Lemma 4.19 repeatedly,
\[
\int_{\mathbb{R}^d} \Delta^k f(x) e^{-i\xi \cdot x} \, dm(x) = \int_{\mathbb{R}^d} f(x) \Delta^k e^{-i\xi \cdot x} \, dm(x) = -|\xi|^{2k} \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dm(x) = - (2\pi)^{d/2} |\xi|^{2k} \hat{f}(\xi)
\]
for any \( k \in \mathbb{N} \). Hence
\[
(2\pi)^{d/2} |\hat{f}(\xi)| \leq |\xi|^{-2k} \|\Delta^k f\|_1 \to 0
\]
as \( |\xi| \to \infty \) and \( \hat{f} \in C_0(\mathbb{R}^d) \). Suppose that \( f \in L^1(\mathbb{R}) \) and \( f_k \in C^\infty_c(\mathbb{R}) \) is a sequence such that \( \lim_{k \to \infty} \|f - f_k\|_1 = 0 \), then \( \lim_{k \to \infty} \|\hat{f} - \hat{f}_k\|_{L^\infty} = 0 \). Hence \( \hat{f} \in C_0(\mathbb{R}^d) \) by an application of Proposition ??.

The next two results give a version of Theorem 4.13 where the convergence holds almost everywhere by making use of the Lebesgue differentiation theorem. Recall for \( f \in L^1_{loc}(\mathbb{R}^d) \) that the **Lebesgue set** of \( f \) defined by,
\[
\mathcal{L}(f) := \left\{ x \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f(x)| \, dy = 0 \right\},
\]
has full Lebesgue measure, i.e. \( m(\mathbb{R}^d \setminus \mathcal{L}(f)) = 0 \).

**Proposition 4.21** (Theorem 4.13 continued). Let \( p \in [1, \infty) \), \( \rho > 0 \) and \( \varphi \in L^\infty(\mathbb{R}^d) \) such that \( 0 \leq \varphi \leq C1_{B(0, \rho)} \) for some \( C < \infty \) and \( \int_{\mathbb{R}^d} \varphi(x) \, dx = 1 \).
If \( f \in L^1_{loc}(\mathbb{R}) \), and \( x \in \mathcal{L}(f) \), then
\[
\lim_{t \downarrow 0} (\varphi_t * f)(x) = f(x),
\]
where \( \varphi_t(x) := t^{-d} \varphi(x/t) \). In particular, \( \varphi_t * f \to f \ a.e. \ as \ t \downarrow 0 \).

**Proof.** Notice that \( 0 \leq \varphi_t \leq C t^{-d} 1_{B(0, \rho t)} \) and therefore for \( x \in \mathcal{L}(f) \) we have, using Theorem ??, that
\[
|\varphi_t * f(x) - f(x)| = \left| \int_{\mathbb{R}^d} [f(x) - f(x)] \varphi_t(y) \, dy \right|
\leq \int_{\mathbb{R}^d} |f(x) - f(x)| \varphi_t(y) \, dy
\leq C t^{-d} \int_{B(0, \rho t)} |f(y) - f(x)| \, dy
= C(\rho, d) \frac{1}{|B(0, \rho t)|} \int_{B(0, \rho t)} |f(x) - f(x)| \, dx \to 0 \text{ as } t \downarrow 0.
\]

The following theorem is an extension of Proposition 4.21.

**Theorem 4.22** (*Theorem 4.15 of Folland*). More general version, assume that \( |\varphi(x)| \leq C(1 + |x|)^{-1-(d+\varepsilon)} \) and \( \int_{\mathbb{R}^d} \varphi(x) \, dx = a \). Then for all \( x \in \mathcal{L}(f) \),
\[
\lim_{t \downarrow 0} (\varphi_t * f)(x) = a f(x)
\]
and in fact,
\[
L(x) := \limsup_{t \downarrow 0} \int |f(x) - f(x)| \varphi_t(y) \, dy = 0.
\]

**Proof.** Throughout this proof \( f \in L^1(\mathbb{R}^d) \) and \( x \in \mathcal{L}(f) \) be fixed and for \( b > 0 \) let
\[
\delta(b) := \frac{1}{b^d} \int_{|y| \leq b} |f(x) - f(x)| \, dy.
\]
From the definition if \( \mathcal{L}(f) \) we know that \( \lim_{b \to 0} \delta(b) = 0 \). The remainder of the proof will be broken into a number of steps.
1. For any \( \eta > 0 \),
\[
L(x) = \limsup_{t \downarrow 0} \int_{|y| \leq \eta} |f(x) - f(x)| \varphi_t(y) \, dy
\]
which is seen as follows;
For any \( \rho > 0 \),
\[
\int_{|y| \leq \rho} |f(x - y) - f(x)| |\varphi_t(y)| \, dy 
\leq C t^{-d} \int_{|y| \leq \rho} |f(x - y) - f(x)| \left(1 + \frac{|y|}{\rho}\right)^{-(d+\varepsilon)} \, dy 
\leq C t^{-d} \int_{|y| \leq \rho} |f(x - y) - f(x)| \left(1 + \frac{a}{t}\right)^{-(d+\varepsilon)} \, dy 
\leq C t^{-d} \delta(b) b^d \left(1 + \frac{a}{t}\right)^{-(d+\varepsilon)} 
= C t^{-d} \delta(b) b^d \left(1 + \frac{a}{t}\right)^{-(d+\varepsilon)} t^\varepsilon b^{-\varepsilon} 
= C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(t + \frac{a}{t})^{d+\varepsilon}}.
\]

Taking \( a = \frac{b}{2} \) in this expression shows,
\[
\int_{\frac{2}{3} < |y| \leq b} |f(x - y) - f(x)| \cdot |\varphi_t(y)| \, dy 
\leq C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(t + \frac{a}{t})^{d+\varepsilon}} 
= C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(2t + 1)^{d+\varepsilon}}.
\]

Taking \( b = 2^{-k} \eta \) and summing the result on \( 0 \leq k \leq K - 1 \) shows
\[
\sum_{k=0}^{K-1} \int_{2^{-(k+1)} \eta < |y| \leq 2^{-k} \eta} |f(x - y) - f(x)| |\varphi_t(y)| \, dy 
\leq C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(2t + 1)^{d+\varepsilon}} 
\leq C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(2t + 1)^{d+\varepsilon}} 
\leq C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(2t + 1)^{d+\varepsilon}}.
\]

We now choose \( K \) so that \( 2^{K+1} \eta \sim 1 \) (i.e. \( 2^{-K} \eta \sim t \)) and we have shown,
\[
\int_{2^{-K} \eta < |y| \leq \eta} |f(x - y) - f(x)| |\varphi_t(y)| \, dy 
= \sum_{k=0}^{K-1} \int_{2^{-(k+1)} \eta < |y| \leq 2^{-k} \eta} |f(x - y) - f(x)| |\varphi_t(y)| \, dy 
\leq C \delta(b) \left(\frac{t}{b}\right)^\varepsilon \frac{1}{(2t + 1)^{d+\varepsilon}}.
\]

4. Combining item 2. with \( \rho = 2^{-K} \eta \sim t \) with item 3. shows
\[
\int_{|y| \leq \eta} |f(x - y) - f(x)| |\varphi_t(y)| \, dy \leq C \delta(\eta).
\]

Combining this result with item 1. implies,
\[
L(x) = \limsup_{t \downarrow 0} \int_{|y| \leq \eta} |f(x - y) - f(x)| |\varphi_t(y)| \, dy 
\leq C \delta(\eta) \to 0 \text{ as } \eta \downarrow 0.
\]

\[ \square \]
5

4

Fourier Transform

5.1 Motivation

Our first goal is to motivate the Fourier inversion formula from the inversion formula for Fourier series (see Exercise ?? below for more details). To do this, for $L > 0$, let $H_L := L^2(\mathbb{R}, \pi L]$ be the $L^2$-Hilbert space equipped with the inner product

$$\langle f | g \rangle_L := \frac{1}{2\pi L} \int_{[-\pi L, \pi L]} f(x) \overline{g}(x) \, dx.$$ 

The linear map, $U_L : H_1 \to H_L$ defined by

$$(U_L f)(x) := f(L^{-1}x) \text{ for } f \in H_1$$

is unitary since

$$\|U_L f\|_L^2 = \frac{1}{2\pi L} \int_{[-\pi L, \pi L]} \|f(L^{-1}x)\|^2 \, dx = \frac{1}{2\pi} \int_{[-\pi, \pi]} \|f(\theta)\|^2 \, d\theta = \|f\|_1^2.$$ 

Letting $\varphi_\lambda(x) = e^{ix\lambda}$, we know that $\{\varphi_k\}_{k=-\infty}^\infty$ is an orthonormal basis for $H_1$ and therefore $\{\varphi_k^L := U_L \varphi_k = \varphi_{L^{-1}k}\}_{k=-\infty}^\infty$ is an orthonormal basis for $H_L$.

Suppose, for simplicity, that $f \in C^1_c(\mathbb{R})$. For sufficiently large $L$ we will have for $|x| \leq \pi L$

$$f(x) = \sum_{k \in \mathbb{Z}} \langle f | \varphi_k^L \rangle_L \varphi_k^L(x)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \frac{1}{L} \left( \int_{[-\pi L, \pi L]} f(y) e^{-iky/L} \, dy \right) \varphi_k^L(x)$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \frac{1}{L} \hat{f} \left( \frac{k}{L} \right) e^{ikx/L}$$

(5.1)

where $\hat{f}$ is the Fourier transform of $f$ defined by,

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y) e^{-iy\xi} \, dy.$$ 

Moreover,

$$\|f\|_{L^2(m)}^2 = 2\pi L \langle f | f \rangle_L = 2\pi L \sum_{k \in \mathbb{Z}} \|f| \varphi_k^L \rangle_L^2$$

$$= \frac{1}{2\pi L} \sum_{k \in \mathbb{Z}} \left( \int_{[-\pi L, \pi L]} f(y) e^{-iky/L} \, dy \right)^2$$

$$= \sum_{k \in \mathbb{Z}} \left| \hat{f} \left( \frac{k}{L} \right) \right|^2 \frac{1}{L}.$$ 

(5.2)

Formally passing to the limit in Eqs. (5.1) and (5.2) suggests that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} \hat{f}(\xi) e^{i\xi x} \, d\xi$$

and $\|f\|_{L^2(m)}^2 = \|\hat{f}\|_{L^2(m)}^2$

which leads one to suspect that the Fourier transform, $f \to \hat{f}$, is a unitary operator on $L^2(\mathbb{R})$. We will eventually show this is the case after first showing how to interpret $\hat{f}$ for $f \in L^2(\mathbb{R})$.

Exercise 5.1 (Wirtinger’s inequality, Folland 8.18). Given $a > 0$ and $f \in C^1([0, a], \mathbb{C})$ such that $f(0) = f(a) = 0$, show$^1$

$$\int_0^a |f(x)|^2 \, dx \leq \left( \frac{a}{\pi} \right)^2 \int_0^a |f'(x)|^2 \, dx.$$ 

Hint: to use the notation above, let $\pi L = a$ and extend $f$ to $[-a, 0]$ by setting $f(-x) = -f(x)$ for $0 \leq x \leq a$. Now compute $\int_0^a |f(x)|^2 \, dx$ and $\int_0^a |f'(x)|^2 \, dx$ in terms of their Fourier coefficients, $\langle f | \varphi_k^L \rangle_L$ and $\langle f' | \varphi_k^L \rangle_L$ respectively.

We now generalize to the $d$-dimensional case. The underlying space in this section is $\mathbb{R}^d$ with Lebesgue measure. As suggested above, the Fourier inversion formula is going to state that

$$f(x) = \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} d\xi e^{i\xi \cdot x} \left[ \int_{\mathbb{R}^d} dy f(y) e^{-i\xi \cdot y} \right].$$ 

(5.3)

If we let $\xi = 2\pi \eta$, this may be written as

$^1$ This inequality is sharp as is seen by taking $f(x) = \sin (\pi x/a)$. 

\[ f(x) = \int_{\mathbb{R}^d} d\eta e^{i2\pi \eta \cdot x} \int_{\mathbb{R}^d} dy f(y) e^{-i2\pi y \cdot \eta} \]

and we have removed the multiplicative factor of \( \left( \frac{1}{2\pi} \right)^d \) in Eq. (5.3) at the expense of placing factors of \( 2\pi \) in the arguments of the exponentials. [This is what Folland does.] Another way to avoid writing the \( 2\pi \)'s altogether is to redefine \( dx \) and \( d\xi \) and this is what we will do here.

**Notation 5.1** Let \( m \) be Lebesgue measure on \( \mathbb{R}^d \) and define:

\[
d\lambda(x) := d\lambda := \left( \frac{1}{\sqrt{2\pi}} \right)^d dm(x) \quad \text{and} \quad d\xi := \left( \frac{1}{\sqrt{2\pi}} \right)^d dm(\xi).
\]

To be consistent with this new normalization of Lebesgue measure we will redefine \( \|f\|_p \), and \( \langle f, g \rangle \), as

\[
\|f\|_p = \left( \int_{\mathbb{R}^d} |f(x)|^p \ d\lambda \right)^{1/p} = \left( \left( \frac{1}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} |f(x)|^p \ dm(x) \right)^{1/p},
\]

and

\[
\langle f, g \rangle := \int_{\mathbb{R}^d} f(x) g(x) \ d\lambda \quad \text{when} \ fg \in L^1.
\]

We also define

\[
\langle f|g \rangle = \langle f, \bar{g} \rangle = \int_{\mathbb{R}^d} f(x) \bar{g}(x) \ d\lambda \quad \text{when} \ fg \in L^1
\]

and a renormalized convolution by \( f \star g := \left( \frac{1}{2\pi} \right)^{d/2} f * g \), i.e.

\[
f \star g(x) = \int_{\mathbb{R}^d} f(x-y)g(y) \ d\lambda = \int_{\mathbb{R}^d} f(x-y)g(y) \left( \frac{1}{2\pi} \right)^{d/2} \ dm(y).
\]

The following notation will also be convenient; given a multi-index \( \alpha \in \mathbb{Z}_+^d \), let \( |\alpha| = \alpha_1 + \cdots + \alpha_d \),

\[
x^{\alpha} := \prod_{j=1}^d x_j^{\alpha_j}, \quad \partial_x^\alpha := \prod_{j=1}^d \left( \frac{\partial}{\partial x_j} \right)^{\alpha_j} \quad \text{and}
\]

\[
D_x^\alpha = \left( \frac{1}{i} \right)^{|\alpha|} \left( \frac{\partial}{\partial x} \right)^\alpha = \left( \frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha.
\]

When \( x \in \mathbb{R}^d \) we let \( |x| = \sqrt{\sum_{j=1}^d x_j^2} \) (which is inconsistent with \( |\alpha| \) for \( \alpha \in \mathbb{Z}_+^d \)) and further let

\[
\langle x \rangle := (1 + |x|^2)^{1/2} \quad \text{and} \quad \nu_s(x) = (1 + |x|)^s \quad \text{for} \ s \in \mathbb{R}.
\]