Part I

240B Homework Exercises (Winter 2012)

The problems below either come from the lecture notes or from Folland. At the moment in the lecture notes please use the following translation table for this course:

a) Random variable = Measurable function on some measure space.
b) Probability measure is a measure where the total measure of the space is 1.
c) $\mathbb{E}[X]$ or $\mathbb{E}X$ is short hand for

$$\int_{\Omega} X(\omega)\, d\mu(\omega),$$

i.e. it is simply shorthand notation for the integral of X relative to a measure, in this case the measure is assumed to μ and the underlying set is Ω.

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HomeWork # 1 Due Friday January 13, 2012

Exercise 1.1 (Measurability). Please review or read Lemma 9.3 of the lecture notes. I will likely use this lemma without mention throughout the course.

Exercise 1.2 (Differentiating past the integral). Please review (or read if it is new to you) Corollary 10.30, Corollary 10.31, and Proposition 10.33 in the lecture notes (Ver. 1). Also review Section 10.3 of the lecture notes (Ver. 1).

Exercise 1.3 (Folland 2.28 on p. 60.). Compute the following limits and justify your calculations:

1. $\lim_{n\to\infty} \int_0^\infty \frac{\sin(\frac{x}{n})}{(1+\frac{x}{n})^n}\, dx$.
2. $\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\, dx$
3. $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\, dx$
4. For all $a \in \mathbb{R}$ compute,
$$f(a) := \lim_{n\to\infty} \int_a^\infty n(1+n^2x^2)^{-1} dx.$$

[Hints: for parts 1. and 2. you might use the binomial expansion to estimate the denominators.]

Exercise 1.4. Let $\Omega := \{1,2,3,4\}$ and $\mathbb{M} := \{1_A, 1_B\}$ where $A := \{1,2\}$ and $B := \{2,3\}$.

a) Show $\sigma(\mathbb{M}) = 2^\Omega$.
b) Find two distinct probability measures, μ and ν on 2^Ω such that $\mu(A) = \nu(A)$ and $\mu(B) = \nu(B)$, i.e.

$$\int_\Omega f d\mu = \int_\Omega f d\nu \tag{1.1}$$

holds for all $f \in \mathbb{M}$.

Moral: the assumption that $\mathbb{M}$ is multiplicative can not be dropped from multiplicative system theorem.

Exercise 1.5. Suppose that $u : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that $\dot{u}(t) \geq 0$ for all t and $\lim_{t\to\pm\infty} u(t) = \pm\infty$. Show that

$$\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} f(u(t))\, \dot{u}(t)\, dt \tag{1.2}$$

for all measurable functions $f : \mathbb{R} \to [0,\infty]$. In particular applying this result to $u(t) = at + b$ where $a > 0$ implies,

$$\int_{\mathbb{R}} f(x)\, dx = a \int_{\mathbb{R}} f(at+b)\, dt.$$

Exercise 1.6. Let $(\Omega, \mathcal{B}, P)$ be a probability space and $X, Y : \Omega \to \mathbb{R}$ be a pair of random variables such that

$$\mathbb{E}\left[f(X)\, g(Y)\right] = \mathbb{E}\left[f(X)\, g(X)\right]$$

for every pair of bounded measurable functions, $f, g : \mathbb{R} \to \mathbb{R}$. Show $P(X = Y) = 1$. **Hint:** Let $\mathbb{H}$ denote the bounded Borel measurable functions, $h : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbb{E}\left[h(X, Y)\right] = \mathbb{E}\left[h(X, X)\right].$$

Use the multiplicative systems Theorem **??** to show $\mathbb{H}$ is the vector space of all bounded Borel measurable functions. Then take $h(x, y) = 1_{\{x=y\}}$.

Exercise 1.7 (Density of $\mathcal{A}$ – simple functions). Let $(\Omega, \mathcal{B}, P)$ be a probability space and assume that $\mathcal{A}$ is a sub-algebra of $\mathcal{B}$ such that $\mathcal{B} = \sigma(\mathcal{A})$. Let $\mathbb{H}$ denote the bounded measurable functions $f : \Omega \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists an an $\mathcal{A}$ – simple function[1], $\varphi : \Omega \to \mathbb{R}$ such that $\mathbb{E}\left|f - \varphi\right| < \varepsilon$. Show $\mathbb{H}$ consists of all bounded measurable functions, $f : \Omega \to \mathbb{R}$. **Hint:** let $\mathbb{M}$ denote the collection of $\mathcal{A}$ – simple functions.

Exercise 1.8 (Density of $\mathcal{A}$ in $\mathcal{B} = \sigma(\mathcal{A})$ – NOT to be collected). Keeping the same notation as in Exercise 1.7 but now take $f = 1_B$ for some $B \in \mathcal{B}$ and given $\varepsilon > 0$, write $\varphi = \sum_{i=0}^n \lambda_i 1_{A_i}$ where $\lambda_0 = 0$, $\{\lambda_i\}_{i=1}^n$ is an enumeration of $\varphi(\Omega) \setminus \{0\}$, and $A_i := \{\varphi = \lambda_i\}$. Show; 1.

$$\mathbb{E}\left|1_B - \varphi\right| = P(A_0 \cap B) + \sum_{i=1}^n \left[\left|1 - \lambda_i\right| P(B \cap A_i) + \left|\lambda_i\right| P(A_i \setminus B)\right] \tag{1.3}$$

$$\geq P(A_0 \cap B) + \sum_{i=1}^n \min\left\{P(B \cap A_i), P(A_i \setminus B)\right\}. \tag{1.4}$$

2. Now let $\psi = \sum_{i=0}^n \alpha_i 1_{A_i}$ with

$$\alpha_i = \begin{cases} 1 \text{ if } P(A_i \setminus B) \leq P(B \cap A_i) \\ 0 \text{ if } P(A_i \setminus B) > P(B \cap A_i) \end{cases}.$$

Then show that

$$\mathbb{E}\left|1_B - \psi\right| = P(A_0 \cap B) + \sum_{i=1}^n \min\left\{P(B \cap A_i), P(A_i \setminus B)\right\} \leq \mathbb{E}\left|1_B - \varphi\right|.$$

Observe that $\psi = 1_D$ where $D = \cup_{i:\alpha_{i=1}} A_i \in \mathcal{A}$ and so you have shown; for every $\varepsilon > 0$ there exists a $D \in \mathcal{A}$ such that

$$P(B \Delta D) = \mathbb{E}\left|1_B - 1_D\right| < \varepsilon.$$

[This problem is the same as one of the propositions in the notes!]

Exercise 1.9. Suppose that $\{(X_i, \mathcal{B}_i)\}_{i=1}^n$ are measurable spaces and for each i, $\mathbb{M}_i$ is a multiplicative system of real bounded measurable functions on X_i such that $\sigma(\mathbb{M}_i) = \mathcal{B}_i$ and there exist $\chi_n \in \mathbb{M}_i$ such that $\chi_n \to 1$ boundedly as $n \to \infty$. Given $f_i : X_i \to \mathbb{R}$ let $f_1 \otimes \cdots \otimes f_n : X_1 \times \cdots \times X_n \to \mathbb{R}$ be defined by

$$(f_1 \otimes \cdots \otimes f_n)(x_1, \ldots, x_n) = f_1(x_1) \ldots f_n(x_n).$$

Show

$$\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n := \{f_1 \otimes \cdots \otimes f_n : f_i \in \mathbb{M}_i \text{ for } 1 \leq i \leq n\}$$

is a multiplicative system of bounded measurable functions on

$$(X := X_1 \times \cdots \times X_n, \mathcal{B} := \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_n)$$

such that $\sigma(\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_n) = \mathcal{B}$. (It is enough to write your solution in the special case where $n = 2$.)

[1] Recall from Definition **??** than f is an $\mathcal{A}$ – simple function if f is a simple function such that $f^{-1}(\{y\}) \in \mathcal{A}$ for all $y \in \mathbb{R}$.

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HomeWork # 2 Due Friday January 20, 2012

Exercise 2.1. Folland 3.27 on p. 107.

Exercise 2.2 (Look at but do not hand in). Folland 3.28 on p. 107.

Exercise 2.3 (Look at but do not hand in). Folland 3.29 on p. 107.

Exercise 2.4. Folland 3.30 on p. 107.

Exercise 2.5. Folland 3.33 on p. 108.

Exercise 2.6. Folland 3.35 on p. 108.

Exercise 2.7. Folland 3.37 on p. 108.

Exercise 2.8. Folland 3.39 on p. 108.

Exercise 2.9 (Look at but do not hand in.). Folland 3.40 on p. 108.

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HomeWork #3 Due Friday January 27, 2012

Exercise 3.1 (Global Integration by Parts Formula). Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are locally absolutely continuous functions[1] such that $f'g$, fg', and fg are all Lebesgue integrable functions on $\mathbb{R}$. Prove the following integration by parts formula;

$$\int_{\mathbb{R}} f'(x) \cdot g(x)\, dx = -\int_{\mathbb{R}} f(x) \cdot g'(x)\, dx. \tag{3.1}$$

Similarly show that; if $f, g : [0, \infty) \to [0, \infty)$ are locally absolutely continuous functions such that $f'g$, fg', and fg are all Lebesgue integrable functions on $[0, \infty)$, then

$$\int_0^\infty f'(x) \cdot g(x)\, dx = -f(0)\, g(0) - \int_0^\infty f(x) \cdot g'(x)\, dx. \tag{3.2}$$

Outline: 1. First use the theory developed to see that Eq. (3.1) holds if $f(x) = 0$ for $|x| \geq N$ for some $N < \infty$.

2. Let $\psi : \mathbb{R} \to [0, 1]$ be a continuously differentiable function such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$.[2] For any $\varepsilon > 0$ let $\psi_\varepsilon(x) = \psi(\varepsilon x)$ Write out the identity in Eq. (3.1) with $f(x)$ being replaced by $f(x)\, \psi_\varepsilon(x)$.

3. Now use the dominated convergence theorem to pass to the limit as $\varepsilon \downarrow 0$ in the identity you found in step 2.

4. A similar outline works to prove Eq. (3.2).

Exercise 3.2. Let (X, d) be a metric space. Suppose that $\{x_n\}_{n=1}^\infty \subset X$ is a sequence and set $\varepsilon_n := d(x_n, x_{n+1})$. Show that for $m > n$ that

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} \varepsilon_k \leq \sum_{k=n}^{\infty} \varepsilon_k.$$

Conclude from this that if

$$\sum_{k=1}^{\infty} \varepsilon_k = \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$$

then $\{x_n\}_{n=1}^\infty$ is Cauchy. Moreover, show that if $\{x_n\}_{n=1}^\infty$ is a convergent sequence and $x = \lim_{n\to\infty} x_n$ then

$$d(x, x_n) \leq \sum_{k=n}^{\infty} \varepsilon_k.$$

Exercise 3.3. Show that (X, d) is a complete metric space iff every sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\sum_{n=1}^\infty d(x_n, x_{n+1}) < \infty$ is a convergent sequence in X. You may find it useful to prove the following statements in the course of the proof.

[1] This means that f and g restricted to any bounded interval in $\mathbb{R}$ are absolutely continuous on that interval.

[2] You may assume the existence of such a ψ, we will deal with this later.

1. If $\{x_n\}$ is Cauchy sequence, then there is a subsequence $y_j := x_{n_j}$ such that $\sum_{j=1}^{\infty} d(y_{j+1}, y_j) < \infty$.
2. If $\{x_n\}_{n=1}^{\infty}$ is Cauchy and there exists a subsequence $y_j := x_{n_j}$ of $\{x_n\}$ such that $x = \lim_{j\to\infty} y_j$ exists, then $\lim_{n\to\infty} x_n$ also exists and is equal to x.

Exercise 3.4. Suppose that $f : [0,\infty) \to [0,\infty)$ is a C^2 – function such that $f(0) = 0$, $f' > 0$ and $f'' \le 0$ and (X, ρ) is a metric space. Show that $d(x, y) = f(\rho(x, y))$ is a metric on X. In particular show that

$$d(x, y) := \frac{\rho(x, y)}{1 + \rho(x, y)}$$

is a metric on X. (Hint: use calculus to verify that $f(a + b) \le f(a) + f(b)$ for all $a, b \in [0,\infty)$.)

Exercise 3.5. Let $\{(X_n, d_n)\}_{n=1}^{\infty}$ be a sequence of metric spaces, $X := \prod_{n=1}^{\infty} X_n$, and for $x = (x(n))_{n=1}^{\infty}$ and $y = (y(n))_{n=1}^{\infty}$ in X let

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x(n), y(n))}{1 + d_n(x(n), y(n))}. \tag{3.3}$$

Show:

1. (X, d) is a metric space,
2. a sequence $\{x_k\}_{k=1}^{\infty} \subset X$ converges to $x \in X$ iff $x_k(n) \to x(n) \in X_n$ as $k \to \infty$ for each $n \in \mathbb{N}$ and
3. X is complete if X_n is complete for all n.

Exercise 3.6 (Look at but do not hand in). Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Show the map

$$(\lambda, x, y) \in \mathbb{F} \times X \times X \to x + \lambda y \in X$$

is continuous relative to the norm on $\mathbb{F} \times X \times X$ defined by

$$\|(\lambda, x, y)\|_{\mathbb{F}\times X\times X} := |\lambda| + \|x\| + \|y\|.$$

Exercise 3.7. Let $X = \mathbb{N}$ and for $p, q \in [1,\infty)$ let $\|\cdot\|_p$ denote the $\ell^p(\mathbb{N})$ – norm. Show $\|\cdot\|_p$ and $\|\cdot\|_q$ are inequivalent norms for $p \ne q$ by showing

$$\sup_{f\ne 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.$$

Exercise 3.8 (Look at but do not hand in.). Let $d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0,\infty)$ be defined by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n},$$

where $\|f\|_n := \sup\{|f(x)| : |x| \le n\} = \max\{|f(x)| : |x| \le n\}$.

1. Show that d is a metric on $C(\mathbb{R})$.
2. Show that a sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ as $n \to \infty$ iff f_n converges to f uniformly on bounded subsets of $\mathbb{R}$.
3. Show that $(C(\mathbb{R}), d)$ is a complete metric space.

[The solution to this exercise is similar to the solution to Exercise 3.5.]

Exercise 3.9. Let $X = C([0, 1], \mathbb{R})$ and for $f \in X$, let

$$\|f\|_1 := \int_0^1 |f(t)|\, dt.$$

Show that $(X, \|\cdot\|_1)$ is normed space and show by example that this space is **not** complete.

Exercise 3.10. By making the change of variables, $u = \ln x$, prove the following facts:

$$\int_0^{1/2} x^{-a} |\ln x|^b \, dx < \infty \iff a < 1 \text{ or } a = 1 \text{ and } b < -1$$
$$\int_2^{\infty} x^{-a} |\ln x|^b \, dx < \infty \iff a > 1 \text{ or } a = 1 \text{ and } b < -1$$
$$\int_0^{1} x^{-a} |\ln x|^b \, dx < \infty \iff a < 1 \text{ and } b > -1$$
$$\int_1^{\infty} x^{-a} |\ln x|^b \, dx < \infty \iff a > 1 \text{ and } b > -1.$$

Suppose $0 < p_0 < p_1 \leq \infty$ and m is Lebesgue measure on $(0, \infty)$. Use the above results to manufacture a function f on $(0, \infty)$ such that $f \in L^p((0, \infty), m)$ iff (a) $p \in (p_0, p_1)$, (b) $p \in [p_0, p_1]$ and (c) $p = p_0$.

Exercise 3.11. Let $(X, \mathcal{B}, \mu)$ be a measure space and $g \in L^1(\mu)$. Show that $\left|\int_X g d\mu\right| = \|g\|_1$ iff there exists $z \in \mathbb{C}$ with $|z| = 1$ such that $g = |g| z$ a.e. (This may be equivalently stated as $\operatorname{sgn}(g(x)) := g(x) / |g(x)|$ is constant for μ -a.e. on the set where $g \neq 0$.) In particular for $f \in L^p(\mu)$ and $g \in L^q(\mu)$ with $q = p/(p-1)$, we have,

$$\left|\int_X fg d\mu\right| = \int_X |fg| \, d\mu \iff \operatorname{sgn}(f(x)) = \overline{\operatorname{sgn}(g(x))} \text{ for } \mu\text{–a.e. } x \in \{fg \neq 0\}.$$

Exercise 3.12. Let $(X, \|\cdot\|_1)$ be the normed space in Exercise 3.9. Compute the closure of A when

1. $A = \{f \in X : f(1/2) = 0\}$.
2. $A = \left\{f \in X : \sup_{t \in [0,1]} f(t) \leq 5\right\}$. [Hint: you may use without proof that if $f_n \in A$ converges to $f \in X$ in $\|\cdot\|_1$ then there is a subsequence which converges for a.e. t.]
3. $A = \left\{f \in X : \int_0^{1/2} f(t) \, dt = 0\right\}$.

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HomeWork #4 Due Friday February 3, 2012

Definition 4.1. *Given a set A contained in a metric space X, let $\bar{A} \subset X$ be the* ***closure of*** *A defined by*

$$\bar{A} := \{x \in X : \exists \{x_n\} \subset A \ni x = \lim_{n\to\infty} x_n\}.$$

That is to say $\bar{A}$ contains all ***limit points*** *of A. We say A is* ***dense in*** *X if $\bar{A} = X$, i.e. every element $x \in X$ is a limit of a sequence of elements from A. A metric space is said to be* ***separable*** *if it contains a countable dense subset, D.*

Exercise 4.1. Given $A \subset X$, show $\bar{A}$ is a closed set and in fact

$$\bar{A} = \cap\{F : A \subset F \subset X \text{ with } F \text{ closed}\}. \tag{4.1}$$

That is to say $\bar{A}$ is the smallest closed set containing A.

Exercise 4.2. If D is a dense subset of a metric space (X, d) and $E \subset X$ is a subset such that to every point $x \in D$ there exists $\{x_n\}_{n=1}^{\infty} \subset E$ with $x = \lim_{n\to\infty} x_n$, then E is also a dense subset of X. If points in E well approximate every point in D and the points in D well approximate the points in X, then the points in E also well approximate all points in X.

Exercise 4.3 (Look at but do not hand in). Suppose that $(X, \|\cdot\|)$ is a normed space and $S \subset X$ is a linear subspace.

1. Show the closure $\bar{S}$ of S is also a linear subspace.
2. Now suppose that X is a Banach space. Show that S with the inherited norm from X is a Banach space iff S is closed.

Exercise 4.4. Let $Y = BC(\mathbb{R}, \mathbb{C})$ be the Banach space of continuous bounded complex valued functions on $\mathbb{R}$ equipped with the uniform norm, $\|f\|_u := \sup_{x\in\mathbb{R}} |f(x)|$. Further let $C_0(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ such that vanish at infinity, i.e. $\lim_{x\to\pm\infty} f(x) = 0$. Also let $C_c(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ with compact support, i.e. there exists $N < \infty$ such that $f(x) = 0$ if $|x| \geq N$. Show $C_0(\mathbb{R}, \mathbb{C})$ is a closed subspace of Y and that

$$\overline{C_c(\mathbb{R}, \mathbb{C})} = C_0(\mathbb{R}, \mathbb{C}).$$

Exercise 4.5. Let $(X, \|\cdot\|)$ be a Banach space. To each $A \in L(X)$, we may define $L_A, R_A : L(X) \to L(X)$ by

$$L_A B = AB \text{ and } R_A B = BA \text{ for all } B \in L(X).$$

Show $L_A, R_A \in L(L(X))$ and that

$$\|L_A\|_{L(L(X))} = \|A\|_{L(X)} = \|R_A\|_{L(L(X))}.$$

For the next three exercises, let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ and $T : X \to Y$ be a linear transformation so that T is given by matrix multiplication by an $m \times n$ matrix. Let us identify the linear transformation T with this matrix.

Exercise 4.6. Assume the norms on X and Y are the ℓ^1 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^n |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\le j\le n} \sum_{i=1}^m |T_{ij}|.$$

Exercise 4.7. Assume the norms on X and Y are the ℓ^∞ – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \max_{1\le j\le n} |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\le i\le m} \sum_{j=1}^n |T_{ij}|.$$

Exercise 4.8. Assume the norms on X and Y are the ℓ^2 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\|^2 = \sum_{j=1}^n x_j^2$. Show $\|T\|^2$ is the largest eigenvalue of the matrix $T^{tr}T : \mathbb{R}^n \to \mathbb{R}^n$. **Hint:** Use the spectral theorem for symmetric real matrices.

Lemma 4.2. *Let X, Y and Z be normed spaces. Then the maps*

$$(S, x) \in L(X, Y) \times X \longrightarrow Sx \in Y$$

and

$$(S, T) \in L(X, Y) \times L(Y, Z) \longrightarrow TS \in L(X, Z)$$

are continuous relative to the norms

$$\|(S, x)\|_{L(X,Y)\times X} := \|S\|_{L(X,Y)} + \|x\|_X \text{ and}$$
$$\|(S, T)\|_{L(X,Y)\times L(Y,Z)} := \|S\|_{L(X,Y)} + \|T\|_{L(Y,Z)}$$

on $L(X, Y) \times X$ and $L(X, Y) \times L(Y, Z)$ respectively.

Exercise 4.9. Suppose that X, Y and Z are Banach spaces and $Q : X \times Y \to Z$ is a bilinear form, i.e. we are assuming $x \in X \to Q(x, y) \in Z$ is linear for each $y \in Y$ and $y \in Y \to Q(x, y) \in Z$ is linear for each $x \in X$. Show Q is continuous relative to the product norm, $\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y$, on $X \times Y$ iff there is a constant $M < \infty$ such that

$$\|Q(x, y)\|_Z \le M \|x\|_X \cdot \|y\|_Y \text{ for all } (x, y) \in X \times Y. \tag{4.2}$$

Then apply this result to prove Lemma 4.2.

Exercise 4.10. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ – finite measure spaces, $f \in L^2(\nu)$ and $k \in L^2(\mu \otimes \nu)$. Show

$$\int |k(x, y)f(y)|\, d\nu(y) < \infty \text{ for } \mu - \text{a.e. } x \tag{4.3}$$

and define

$$Kf(x) := \int_Y k(x, y)f(y)d\nu(y) \tag{4.4}$$

when the integral is defined and set $Kf(x) = 0$ otherwise. Show $Kf \in L^2(\mu)$ and $K : L^2(\nu) \to L^2(\mu)$ is a bounded operator with $\|K\|_{op} \le \|k\|_{L^2(\mu\otimes\nu)}$. [Remember that $L^2(\nu)$ and $L^2(\mu)$ consists of equivalence classes of functions.]

Theorem 4.3 (B. L. T. Theorem). *Suppose that Z is a normed space, X is a Banach space, and $\mathcal{S} \subset Z$ is a dense linear subspace of Z. If $T : \mathcal{S} \to X$ is a bounded linear transformation (i.e. there exists $C < \infty$ such that $\|Tz\| \le C\|z\|$ for all $z \in \mathcal{S}$), then T has a unique extension to an element $\bar{T} \in L(Z, X)$ and this extension still satisfies*

$$\|\bar{T}z\| \le C\|z\| \text{ for all } z \in \bar{\mathcal{S}}.$$

Exercise 4.11 (Look at but do not hand in). Prove Theorem 4.3.

5

HomeWork #5 Due Friday February 10, 2012

Exercise 5.1. Suppose (X, d) is a metric space which contains an uncountable subset $\Lambda \subset X$ with the property that there exists $\varepsilon > 0$ such that $d(a, b) \geq \varepsilon$ for all $a, b \in \Lambda$ with $a \neq b$. Show that (X, d) is **not** separable.

Exercise 5.2. Show $\ell^\infty(\mathbb{N})$ is not separable. Hint: find an uncountable set $\Lambda \subset \ell^\infty(\mathbb{N})$ as in the hypothesis of Exercise 5.1.

Exercise 5.3. Suppose M is a subset of H, then $M^{\perp\perp} = \overline{\operatorname{span}(M)}$ where (as usual), $\operatorname{span}(M)$ denotes all finite linear combinations of elements from M.

Exercise 5.4. Let H, K, M be Hilbert spaces, $A, B \in L(H, K)$, $C \in L(K, M)$ and $\lambda \in \mathbb{C}$. Show $(A + \lambda B)^* = A^* + \bar{\lambda} B^*$ and $(CA)^* = A^* C^* \in L(M, H)$.

Exercise 5.5. Let $H = \mathbb{C}^n$ and $K = \mathbb{C}^m$ equipped with the usual inner products, i.e. $\langle z|w\rangle_H = z \cdot \bar{w}$ for $z, w \in H$. Let A be an $m \times n$ matrix thought of as a linear operator from H to K. Show the matrix associated to $A^* : K \to H$ is the conjugate transpose of A.

Remark 5.1 (Polarization identity). It is sometimes useful to know that the inner product $\langle \cdot | \cdot \rangle$ may be reconstructed from knowledge of its associated norm $\|\cdot\|$. For example in the real case we have

$$\langle x|y\rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right). \tag{5.1}$$

Similarly if we are working over $\mathbb{C}$, then by direct computation,

$$2\operatorname{Re}\langle x|y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2$$

and

$$-2\operatorname{Re}\langle x|y\rangle = \|x-y\|^2 - \|x\|^2 - \|y\|^2.$$

Subtracting these two equations gives the "polarization identity,"

$$4\operatorname{Re}\langle x|y\rangle = \|x+y\|^2 - \|x-y\|^2.$$

Replacing y by iy in this equation then implies that

$$4\operatorname{Im}\langle x|y\rangle = \|x+iy\|^2 - \|x-iy\|^2$$

from which we find

$$\langle x|y\rangle = \frac{1}{4}\sum_{\varepsilon \in G} \varepsilon \|x + \varepsilon y\|^2 \tag{5.2}$$

where $G = \{\pm 1, \pm i\}$ – a cyclic subgroup of $S^1 \subset \mathbb{C}$.

Exercise 5.6. Let $U : H \to K$ be a linear map between complex Hilbert spaces. Show the following are equivalent:

1. $U : H \to K$ is an isometry,
2. $\langle Ux|Ux'\rangle_K = \langle x|x'\rangle_H$ for all $x, x' \in H$,
3. $U^*U = id_H$.

Hint: use the polarization identity in Eq. (5.2) of Remark 5.1.

Exercise 5.7. Let $U : H \to K$ be a linear map, show the following are equivalent:

1. $U : H \to K$ is unitary
2. $U^*U = id_H$ and $UU^* = id_K$.
3. U is invertible and $U^{-1} = U^*$.

Definition 5.2 (Strong Convergence). *Let X be a Banach space. We say a sequence of operators $\{A_n\}_{n=1}^\infty \subset L(X)$* ***converges strongly*** *to $A \in L(X)$ if $\lim_{n\to\infty} A_n x = Ax$ for all $x \in X$. We abbreviate this by [**Note well** that strong convergence is weaker than norm convergence, i.e. if $\|A - A_n\|_{op} \to 0$ then $A_n \xrightarrow{s} A$ as $n \to \infty$ but no necessarily the other way around unless $\dim X < \infty$.]*

Exercise 5.8 (Strong convergence of increasing projections). Let $(H, \langle\cdot|\cdot\rangle)$ be a Hilbert space and suppose that $\{P_n\}_{n=1}^\infty$ is a sequence of orthogonal projection operators on H such that $P_n(H) \subset P_{n+1}(H)$ for all n. Let $M := \cup_{n=1}^\infty P_n(H)$ (a subspace of H) and let P denote orthonormal projection onto $\bar{M}$. Show $P_n \xrightarrow{s} P$ as $n \to \infty$, i.e. $\lim_{n\to\infty} P_n x = Px$ for all $x \in H$. **Hint:** first prove the result for $x \in M^\perp$, then for $x \in M$ and then for $x \in \bar{M}$.

Exercise 5.9 (The Mean Ergodic Theorem). Let $U : H \to H$ be a unitary operator on a Hilbert space H, $M = \mathrm{Nul}(U - I)$, $P = P_M$ be orthogonal projection onto M, and $S_n = \frac{1}{n}\sum_{k=0}^{n-1} U^k$. Show $S_n \xrightarrow{s} P_M$ as $n \to \infty$, i.e. $\lim_{n\to\infty} S_n x = P_M x$ for all $x \in H$.

Hints: 1. Show H is the orthogonal direct sum of M and $\overline{\mathrm{Ran}(U - I)}$ by using a Lemma from the notes after first showing $\mathrm{Nul}(U^* - I) = \mathrm{Nul}(U - I)$. 2. Verify the result for $x \in \mathrm{Nul}(U - I)$ and $x \in \mathrm{Ran}(U - I)$. 3. Use a limiting argument to verify the result for $x \in \overline{\mathrm{Ran}(U - I)}$.

5.1 Extra problems to ponder – do not hand in.

Exercise 5.10. Let $(X, \mathcal{B}, \mu)$ be a σ – finite measure space. Suppose that $1 \le p < \infty$ and to each $\varphi \in L^\infty(\mu)$ show $M_\varphi \in L(L^p(\mu))$ be defined by $M_\varphi f = \varphi f$ for all $f \in L^p(\mu)$ – so M_φ is multiplication by φ. Show $\|M_\varphi\|_{op} = \|\varphi\|_\infty$, i.e. $L^\infty(\mu) \ni \varphi \to M_\varphi \in L(L^p(\mu))$ is an isometry.

Exercise 5.11. Let m be Lebesgue measure on $([0,\infty), \mathcal{B}, m)$.

1. Show $L^\infty(([0,\infty), \mathcal{B}, m))$ is not separable. Hint:you might produce an isometry from $\ell^\infty(\mathbb{N})$ into $L^\infty([0,\infty), m)$ and then use Exercise 5.2. find an uncountable set $\Lambda \subset L^\infty(\mathbb{R}, m)$ as in the hypothesis of Exercise 5.1.
2. Use this result along with Exercise 5.10 in order to show $L(L^p(([0,\infty), \mathcal{B}, m)))$ is not separable for all $1 \le p < \infty$.

Exercise 5.12 (A "Martingale" Convergence Theorem). Suppose that $\{M_n\}_{n=1}^\infty$ is an increasing sequence of closed subspaces of a Hilbert space, H, $P_n := P_{M_n}$, and $\{x_n\}_{n=1}^\infty$ is a sequence of elements from H such that $x_n = P_n x_{n+1}$ for all $n \in \mathbb{N}$. Show;

1. $P_m x_n = x_m$ for all $1 \le m \le n < \infty$,
2. $(x_n - x_m) \perp M_m$ for all $n \ge m$,
3. $\|x_n\|$ is increasing as n increases,
4. if $\sup_n \|x_n\| = \lim_{n\to\infty} \|x_n\| < \infty$, then $x := \lim_{n\to\infty} x_n$ exists in M and that $x_n = P_n x$ for all $n \in \mathbb{N}$. (**Hint:** show $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence.)

Exercise 5.13. Let H be a Hilbert space. Use Theorem about orthonormal bases to show there exists a set X and a unitary map $U : H \to \ell^2(X)$. Moreover, if H is separable and $\dim(H) = \infty$, then X can be taken to be $\mathbb{N}$ so that H is unitarily equivalent to $\ell^2 = \ell^2(\mathbb{N})$.

Remark 5.3. Let $H = \ell^2 := L^2(\mathbb{N}, \text{counting measure})$,

$$M_n = \{(a(1), \dots, a(n), 0, 0, \dots) : a(i) \in \mathbb{C} \text{ for } 1 \le i \le n\},$$

and $x_n(i) = 1_{i \le n}$, then $x_m = P_m x_n$ for all $n \ge m$ while $\|x_n\|^2 = n \uparrow \infty$ as $n \to \infty$. Thus, we can not drop the assumption that $\sup_n \|x_n\| < \infty$ in Exercise 5.12.

6

HomeWork #6 Due Friday February 24, 2012

Exercise 6.1. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces such that $L^2(\mu)$ and $L^2(\nu)$ are separable. For $f \in L^2(\mu)$ and $g \in L^2(\nu)$ let $(f \otimes g)(x, y) := f(x) g(y)$ so that $f \otimes g \in L^2(\mu \otimes \nu)$. If $\{f_n\}_{n=1}^\infty$ and $\{g_m\}_{m=1}^\infty$ are orthonormal bases for $L^2(\mu)$ and $L^2(\nu)$ respectively, show $\beta := \{f_n \otimes g_m : m, n \in \mathbb{N}\}$ is an orthonormal basis for $L^2(\mu \otimes \nu)$. **Hint:** you might model your proof on the discrete analogue of this theorem discussed in the notes.

6.1 Fourier Series Exercises

Exercise 6.2. Show that if $f \in L^1\left([-\pi, \pi]^d\right)$ and $\tilde{f}(k) = 0$ for all k then $f = 0$ a.e.

Exercise 6.3. Show $\sum_{k=1}^\infty k^{-2} = \pi^2/6$, by taking $f(x) = x$ on $[-\pi, \pi]$ and computing $\|f\|_2^2$ directly and then in terms of the Fourier Coefficients $\tilde{f}$ of f.

Exercise 6.4 (Riemann Lebesgue Lemma for Fourier Series). Show for $f \in L^1\left([-\pi, \pi]^d\right)$ that $\tilde{f} \in c_0(\mathbb{Z}^d)$, i.e. $\tilde{f} : \mathbb{Z}^d \to \mathbb{C}$ and $\lim_{k\to\infty} \tilde{f}(k) = 0$. **Hint:** If $f \in L^2\left([-\pi, \pi]^d\right)$, this follows form Bessel's inequality. Now use a density argument.

Exercise 6.5 (Do not hand in). Suppose $f \in L^1([-\pi, \pi]^d)$ is a function such that $\tilde{f} \in \ell^1(\mathbb{Z}^d)$ and set

$$g(x) := \sum_{k \in \mathbb{Z}^d} \tilde{f}(k) e^{ik\cdot x} \text{ (pointwise).}$$

1. Show $g \in C_{per}(\mathbb{R}^d)$.
2. Show $g(x) = f(x)$ for m – a.e. x in $[-\pi, \pi]^d$. **Hint**: Show $\tilde{g}(k) = \tilde{f}(k)$ and apply Exercise 6.2.
3. Conclude that $f \in L^1([-\pi, \pi]^d) \cap L^\infty([-\pi, \pi]^d)$ and in particular $f \in L^p([-\pi, \pi]^d)$ for all $p \in [1, \infty]$.

Notation 6.1 *Given a multi-index $\alpha \in \mathbb{Z}_+^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$,*

$$x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}, \text{ and } \partial_x^\alpha = \left(\frac{\partial}{\partial x}\right)^\alpha := \prod_{j=1}^d \left(\frac{\partial}{\partial x_j}\right)^{\alpha_j}.$$

Further for $k \in \mathbb{N}_0$, let $f \in C_{per}^k(\mathbb{R}^d)$ iff $f \in C^k(\mathbb{R}^d) \cap C_{per}\left(\mathbb{R}^d\right)$, $\partial_x^\alpha f(x)$ exists and is continuous for $|\alpha| \le k$.

Exercise 6.6 (Smoothness implies decay). Suppose $m \in \mathbb{N}_0$, α is a multi-index such that $|\alpha| \le 2m$ and $f \in C_{per}^{2m}(\mathbb{R}^d)$[1].

[1] We view $C_{per}(\mathbb{R}^d)$ as a subspace of $H = L^2\left([-\pi, \pi]^d\right)$ by identifying $f \in C_{per}(\mathbb{R}^d)$ with $f|_{[-\pi,\pi]^d} \in H$.

1. Using integration by parts, show (using Notation 6.1) that

$$(ik)^\alpha \tilde{f}(k) = \langle \partial^\alpha f | \varphi_k \rangle \text{ for all } k \in \mathbb{Z}^d.$$

Note: This equality implies

$$\left|\tilde{f}(k)\right| \le \frac{1}{k^\alpha} \|\partial^\alpha f\|_H \le \frac{1}{k^\alpha} \|\partial^\alpha f\|_\infty .$$

2. Now let $\Delta f = \sum_{i=1}^d \partial^2 f/\partial x_i^2$, Working as in part 1) show

$$\langle (1-\Delta)^m f | \varphi_k \rangle = (1+|k|^2)^m \tilde{f}(k). \tag{6.1}$$

Remark 6.2. Suppose that m is an even integer, α is a multi-index and $f \in C_{per}^{m+|\alpha|}(\mathbb{R}^d)$, then

$$\begin{aligned}
\left(\sum_{k\in\mathbb{Z}^d} |k^\alpha| \left|\tilde{f}(k)\right|\right)^2 &= \left(\sum_{k\in\mathbb{Z}^d} |\langle \partial^\alpha f | e_k\rangle| \, (1+|k|^2)^{m/2} (1+|k|^2)^{-m/2}\right)^2 \\
&= \left(\sum_{k\in\mathbb{Z}^d} \left|\langle (1-\Delta)^{m/2} \partial^\alpha f | e_k\rangle\right| (1+|k|^2)^{-m/2}\right)^2 \\
&\le \sum_{k\in\mathbb{Z}^d} \left|\langle (1-\Delta)^{m/2} \partial^\alpha f | e_k\rangle\right|^2 \cdot \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-m} \\
&= C_m \left\|(1-\Delta)^{m/2} \partial^\alpha f\right\|_H^2
\end{aligned}$$

where $C_m := \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{-m} < \infty$ iff $m > d/2$. So the smoother f is the faster $\tilde{f}$ decays at infinity. The next problem is the converse of this assertion and hence smoothness of f corresponds to decay of $\tilde{f}$ at infinity and visa-versa.

Exercise 6.7 (A Sobolev Imbedding Theorem). Suppose $s \in \mathbb{R}$ and $\left\{c_k \in \mathbb{C} : k \in \mathbb{Z}^d\right\}$ are coefficients such that

$$\sum_{k\in\mathbb{Z}^d} |c_k|^2 (1+|k|^2)^s < \infty.$$

Show if $s > \frac{d}{2} + m$, the function f defined by

$$f(x) = \sum_{k\in\mathbb{Z}^d} c_k e^{ik\cdot x}$$

is in $C_{per}^m(\mathbb{R}^d)$. **Hint**: Work as in the above remark to show

$$\sum_{k\in\mathbb{Z}^d} |c_k| \, |k^\alpha| < \infty \text{ for all } |\alpha| \le m.$$

Exercise 6.8 (Poisson Summation Formula). Let $F \in L^1(\mathbb{R}^d)$,

$$E := \left\{ x \in \mathbb{R}^d : \sum_{k\in\mathbb{Z}^d} |F(x+2\pi k)| = \infty \right\}$$

and set

$$\hat{F}(k) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} F(x) e^{-ik\cdot x} dx \text{ for } k \in \mathbb{Z}^d.$$

Further assume $\hat{F} \in \ell^1(\mathbb{Z}^d)$. [This can be achieved by assuming F is sufficiently differentiable with the derivatives being integrable like in Exercise 6.6.]

1. Show $m(E) = 0$ and $E + 2\pi k = E$ for all $k \in \mathbb{Z}^d$. **Hint**: Compute $\int_{[-\pi,\pi]^d} \sum_{k\in\mathbb{Z}^d} |F(x + 2\pi k)|\, dx$.
2. Let
$$f(x) := \begin{cases} \sum_{k\in\mathbb{Z}^d} F(x + 2\pi k) & \text{for } x \notin E \\ 0 & \text{if } x \in E. \end{cases}$$
Show $f \in L^1([-\pi, \pi]^d)$ and $\tilde{f}(k) = (2\pi)^{-d/2} \hat{F}(k)$.
3. Using item 2) and the assumptions on F, show
$$f(x) = \sum_{k\in\mathbb{Z}^d} \tilde{f}(k)e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^d} (2\pi)^{-d/2} \hat{F}(k)e^{ik\cdot x} \text{ for } m - \text{a.e. } x,$$
i.e.
$$\sum_{k\in\mathbb{Z}^d} F(x + 2\pi k) = (2\pi)^{-d/2} \sum_{k\in\mathbb{Z}^d} \hat{F}(k)e^{ik\cdot x} \text{ for } m - \text{a.e. } x \tag{6.2}$$
and form this conclude that $f \in L^1([-\pi, \pi]^d) \cap L^\infty([-\pi, \pi]^d)$.
Hint: see the hint for item 2. of Exercise 6.5.
4. Suppose we now assume that $F \in C(\mathbb{R}^d)$ and F satisfies $|F(x)| \le C(1 + |x|)^{-s}$ for some $s > d$ and $C < \infty$. Under these added assumptions on F, show Eq. (6.2) holds for **all** $x \in \mathbb{R}^d$ and in particular
$$\sum_{k\in\mathbb{Z}^d} F(2\pi k) = (2\pi)^{-d/2} \sum_{k\in\mathbb{Z}^d} \hat{F}(k).$$

For notational simplicity, in the remaining problems we will assume that $d = 1$.

Exercise 6.9. Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$, then $\mathbb{D} := \text{span}\{e^{i\lambda\cdot x} : \lambda \in \mathbb{R}^d\}$ is a dense subspace of $L^p(\mu)$ for all $1 \le p < \infty$. **Hints:** from class and the notes we know that $C_c(\mathbb{R}^d)$ is a dense subspace of $L^p(\mu)$. For $f \in C_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$, let
$$f_N(x) := \sum_{n\in\mathbb{Z}^d} f(x + 2\pi Nn).$$
Show $f_N \in BC(\mathbb{R}^d)$ and $x \to f_N(Nx)$ is 2π – periodic and so $x \to f_N(Nx)$ can be approximated uniformly by trigonometric polynomials. Use this fact to conclude that $f_N \in \bar{\mathbb{D}}^{L^p(\mu)}$. After this show $f_N \to f$ in $L^p(\mu)$.

Exercise 6.10. Suppose that μ and ν are two finite measures on $\mathbb{R}^d$ such that
$$\int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\mu(x) = \int_{\mathbb{R}^d} e^{i\lambda\cdot x} d\nu(x) \tag{6.3}$$
for all $\lambda \in \mathbb{R}^d$. Show $\mu = \nu$.

Hint: Perhaps the easiest way to do this is to use Exercise 6.9 with the measure μ being replaced by $\mu+\nu$. Alternatively, use the method of proof of Exercise 6.9 to show Eq. (6.3) implies $\int_{\mathbb{R}^d} f d\mu(x) = \int_{\mathbb{R}^d} f d\nu(x)$ for all $f \in C_c(\mathbb{R}^d)$ and then apply Corollary **??**.

Exercise 6.11. Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$. Further assume that $C_M := \int_{\mathbb{R}^d} e^{M|x|} d\mu(x) < \infty$ for all $M \in (0, \infty)$. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of polynomials, $\rho(x) = \sum_{|\alpha|\le N} \rho_\alpha x^\alpha$ with $\rho_\alpha \in \mathbb{C}$, on $\mathbb{R}^d$. (Notice that $|\rho(x)|^p \le Ce^{M|x|}$ for some constant $C = C(\rho, p, M)$, so that $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$.) Show $\mathcal{P}(\mathbb{R}^d)$ is dense in $L^p(\mu)$ for all $1 \le p < \infty$. Here is a possible outline.

Outline: Fix a $\lambda \in \mathbb{R}^d$ and let $f_n(x) = (\lambda \cdot x)^n / n!$ for all $n \in \mathbb{N}$.

1. Use calculus to verify $\sup_{t\ge 0} t^\alpha e^{-Mt} = (\alpha/M)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/M)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-M|x|}\right) |\lambda|^{pn} e^{M|x|}$$
to find an estimate on $\|f_n\|_p$.
2. Use your estimate on $\|f_n\|_p$ to show $\sum_{n=0}^\infty \|f_n\|_p < \infty$ and conclude
$$\lim_{N\to\infty} \left\| e^{i\lambda\cdot(\cdot)} - \sum_{n=0}^N i^n f_n \right\|_p = 0.$$
3. Now finish by appealing to Exercise 6.9.

6.2 Extra Problems to ponder (Do not hand in.)

Exercise 6.12 (Heat Equation 1.). Let $(t,x) \in [0,\infty) \times \mathbb{R} \to u(t,x)$ be a continuous function such that $u(t,\cdot) \in C_{per}(\mathbb{R})$ for all $t \geq 0$, $\dot{u} := u_t$, u_x, and u_{xx} exists and are continuous when $t > 0$. Further assume that u satisfies the heat equation $\dot{u} = \frac{1}{2}u_{xx}$. Let $\tilde{u}(t,k) := \langle u(t,\cdot)|\varphi_k\rangle$ for $k \in \mathbb{Z}$. Show for $t > 0$ and $k \in \mathbb{Z}$ that $\tilde{u}(t,k)$ is differentiable in t and $\frac{d}{dt}\tilde{u}(t,k) = -k^2\tilde{u}(t,k)/2$. Use this result to show for $t > 0$ that

$$u(t,x) = \sum_{k\in\mathbb{Z}} e^{-\frac{t}{2}k^2}\tilde{f}(k)e^{ikx} \tag{6.4}$$

where $f(x) := u(0,x)$ and as above

$$\tilde{f}(k) = \langle f|\varphi_k\rangle = \int_{-\pi}^{\pi} f(y)e^{-iky}dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dm\,(y)\,.$$

Notice from Eq. (6.4) that $(t,x) \to u(t,x)$ is C^∞ for $t > 0$.

Exercise 6.13 (Heat Equation 2.). Let $q_t(x) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{-\frac{t}{2}k^2}e^{ikx}$. Show that Eq. (6.4) may be rewritten as

$$u(t,x) = \int_{-\pi}^{\pi} q_t(x-y)f(y)dy$$

and

$$q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x+k2\pi)$$

where $p_t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}x^2}$. Also show $u(t,x)$ may be written as

$$u(t,x) = p_t * f(x) := \int_{\mathbb{R}^d} p_t(x-y)f(y)dy.$$

Hint: To show $q_t(x) = \sum_{k\in\mathbb{Z}} p_t(x+k2\pi)$, use the Poisson summation formula (Exercise 6.8) and the Gaussian integration identity,

$$\hat{p}_t(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} p_t(x)e^{i\omega x}dx = \frac{1}{\sqrt{2\pi}}e^{-\frac{t}{2}\omega^2}. \tag{6.5}$$

Equation (6.5) will be discussed in more detail later in the class.

Exercise 6.14 (Wave Equation). Let $u \in C^2(\mathbb{R}\times\mathbb{R})$ be such that $u(t,\cdot) \in C_{per}(\mathbb{R})$ for all $t \in \mathbb{R}$. Further assume that u solves the wave equation, $u_{tt} = u_{xx}$. Let $f(x) := u(0,x)$ and $g(x) = \dot{u}(0,x)$. Show $\tilde{u}(t,k) := \langle u(t,\cdot),\varphi_k\rangle$ for $k \in \mathbb{Z}$ is twice continuously differentiable in t and $\frac{d^2}{dt^2}\tilde{u}(t,k) = -k^2\tilde{u}(t,k)$. Use this result to show

$$u(t,x) = \sum_{k\in\mathbb{Z}}\left(\tilde{f}(k)\cos(kt) + \tilde{g}(k)\frac{\sin kt}{k}\right)e^{ikx} \tag{6.6}$$

with the sum converging absolutely. Also show that $u(t,x)$ may be written as

$$u(t,x) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{-t}^{t} g(x+\tau)d\tau. \tag{6.7}$$

Hint: To show Eq. (6.6) implies (6.7) use

$$\cos kt = \frac{e^{ikt}+e^{-ikt}}{2},$$
$$\sin kt = \frac{e^{ikt}-e^{-ikt}}{2i}, \text{ and}$$
$$\frac{e^{ik(x+t)}-e^{ik(x-t)}}{ik} = \int_{-t}^{t} e^{ik(x+\tau)}d\tau.$$

Exercise 6.15. Suppose H is a Hilbert space and $\{H_n : n \in \mathbb{N}\}$ are closed subspaces of H such that $H_n \perp H_m$ for all $m \neq n$ and if $f \in H$ with $f \perp H_n$ for all $n \in \mathbb{N}$, then $f = 0$. For $f \in \oplus_{n=1}^{\infty} H_n$, show the sum $\sum_{n=1}^{\infty} f(n)$ is convergent in H and the map $U : \oplus_{n=1}^{\infty} H_n \to H$ defined by $Uf := \sum_{n=1}^{\infty} f(n)$ is unitary.

Exercise 6.16. Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $X = \coprod_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{M}$ and $\mu(X_n) > 0$ for all n. Then $U : L^2(X, \mu) \to \oplus_{n=1}^{\infty} L^2(X_n, \mu)$ defined by $(Uf)(n) := f 1_{X_n}$ is unitary.

Exercise 6.17 (Knowledge of analytic functions is helpful). Again let μ be a finite measure on $\mathcal{B}_{\mathbb{R}^d}$ but now assume there exists an $\varepsilon > 0$ such that $C := \int_{\mathbb{R}^d} e^{\varepsilon|x|} d\mu(x) < \infty$. Also let $q > 1$ and $h \in L^q(\mu)$ be a function such that $\int_{\mathbb{R}^d} h(x) x^\alpha d\mu(x) = 0$ for all $\alpha \in \mathbb{N}_0^d$. (As mentioned in Exercise 6.17, $\mathcal{P}(\mathbb{R}^d) \subset L^p(\mu)$ for all $1 \le p < \infty$, so $x \to h(x)x^\alpha$ is in $L^1(\mu)$.) Show $h(x) = 0$ for $\mu-$ a.e. x using the following outline.

Outline: Fix a $\lambda \in \mathbb{R}^d$, let $f_n(x) = (\lambda \cdot x)^n / n!$ for all $n \in \mathbb{N}$, and let $p = q/(q-1)$ be the conjugate exponent to q.

1. Use calculus to verify $\sup_{t \ge 0} t^\alpha e^{-\varepsilon t} = (\alpha/\varepsilon)^\alpha e^{-\alpha}$ for all $\alpha \ge 0$ where $(0/\varepsilon)^0 := 1$. Use this estimate along with the identity
$$|\lambda \cdot x|^{pn} \le |\lambda|^{pn} |x|^{pn} = \left(|x|^{pn} e^{-\varepsilon|x|}\right) |\lambda|^{pn} e^{\varepsilon|x|}$$
to find an estimate on $\|f_n\|_p$.
2. Use your estimate on $\|f_n\|_p$ to show there exists $\delta > 0$ such that $\sum_{n=0}^{\infty} \|f_n\|_p < \infty$ when $|\lambda| \le \delta$ and conclude for $|\lambda| \le \delta$ that $e^{i\lambda \cdot x} = L^p(\mu)\text{-}\sum_{n=0}^{\infty} i^n f_n(x)$. Conclude from this that
$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ when } |\lambda| \le \delta.$$
3. Let $\lambda \in \mathbb{R}^d$ ($|\lambda|$ not necessarily small) and set $g(t) := \int_{\mathbb{R}^d} e^{it\lambda \cdot x} h(x) d\mu(x)$ for $t \in \mathbb{R}$. Show $g \in C^\infty(\mathbb{R})$ and
$$g^{(n)}(t) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{it\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$
4. Let $T = \sup\{\tau \ge 0 : g|_{[0,\tau]} \equiv 0\}$. By Step 2., $T \ge \delta$. If $T < \infty$, then
$$0 = g^{(n)}(T) = \int_{\mathbb{R}^d} (i\lambda \cdot x)^n e^{iT\lambda \cdot x} h(x) d\mu(x) \text{ for all } n \in \mathbb{N}.$$
Use Step 3. with h replaced by $e^{iT\lambda \cdot x} h(x)$ to conclude
$$g(T+t) = \int_{\mathbb{R}^d} e^{i(T+t)\lambda \cdot x} h(x) d\mu(x) = 0 \text{ for all } t \le \delta / |\lambda|.$$
This violates the definition of T and therefore $T = \infty$ and in particular we may take $T = 1$ to learn
$$\int_{\mathbb{R}^d} h(x) e^{i\lambda \cdot x} d\mu(x) = 0 \text{ for all } \lambda \in \mathbb{R}^d.$$
5. Use Exercise 6.9 to conclude that
$$\int_{\mathbb{R}^d} h(x) g(x) d\mu(x) = 0$$
for all $g \in L^p(\mu)$. Now choose g judiciously to finish the proof.