Math 241A Homework Problem List for Fall 2015

1.1 Homework 1. Due Wednesday, October 7, 2015

Notation 1.1 *Let U be any set, g be a positive function on U, and Y be a normed space. For any $f : U \to Y$ let*

$$\|f\|_g := \sup_{x \in U} [g(x) \|f(x)\|_Y]$$

and

$$B_g(U, V) := \left\{ f : U \to Y \,|\, \|f\|_g < \infty \right\}.$$

- **Hand in** exercises: 2.1, 2.2
- **Look at** exercises: 2.3

1.2 Homework 2. Due Wednesday, October 14, 2015

- **Hand in** exercises: 2.4, 2.8, 2.9
- **Look at** from lecture note exercises: 2.5, 2.6, 2.7, 2.10, 2.11, 2.12

Problems

Exercise 2.1. Let $Y = BC(\mathbb{R}, \mathbb{C})$ be the Banach space of continuous bounded complex valued functions on $\mathbb{R}$ equipped with the uniform norm, $\|f\|_u := \sup_{x\in\mathbb{R}} |f(x)|$. Further let $C_0(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ such that vanish at infinity, i.e. $\lim_{x\to\pm\infty} f(x) = 0$. Also let $C_c(\mathbb{R}, \mathbb{C})$ denote those $f \in C(\mathbb{R}, \mathbb{C})$ with compact support, i.e. there exists $N < \infty$ such that $f(x) = 0$ if $|x| \geq N$. Show $C_0(\mathbb{R}, \mathbb{C})$ is a closed subspace of Y and that

$$\overline{C_c(\mathbb{R}, \mathbb{C})} = C_0(\mathbb{R}, \mathbb{C}).$$

Exercise 2.2. Let U be any set, g be a positive function on U, and Y be a normed space. For any $f : U \to Y$ let

$$\|f\|_g := \sup_{x\in U} [g(x) \|f(x)\|_Y]$$

and let

$$B_g(U, V) := \left\{ f : U \to Y \,|\, \|f\|_g < \infty \right\}.$$

Verify that $\|\cdot\|_g$ makes $B_g(U, Y)$ into a normed space. Moreover show; if Y is a Banach space, then $\left(B_g(U, Y), \|\cdot\|_g\right)$ is a Banach space.

Exercise 2.3. Now assume that U is a subset of $\mathbb{R}^n$ (or more generally any topological space) and $g \in C(U, (0, \infty))$. Show

$$B_g C(U, Y) := B_g(U, Y) \cap C(U, Y)$$

is a closed subspace of $B_g(U, Y)$. In particular if Y is a Banach space, then $B_g C(U, Y)$ is a also a Banach space.

Exercise 2.4 (Prove the BLT Theorem). Suppose that Z is a normed space, X is a Banach space, and $\mathcal{S} \subset Z$ is a dense linear subspace of Z. If $T : \mathcal{S} \to X$ is a bounded linear transformation (i.e. there exists $C < \infty$ such that $\|Tz\| \leq C\|z\|$ for all $z \in \mathcal{S}$), then T has a unique extension to an element $\bar{T} \in L(Z, X)$ and this extension still satisfies

$$\|\bar{T}z\| \leq C\|z\| \text{ for all } z \in \bar{\mathcal{S}}.$$

Exercise 2.5. Assume the norms on X and Y are the ℓ^1 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^n |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq j\leq n} \sum_{i=1}^m |T_{ij}|.$$

Exercise 2.6. Assume the norms on X and Y are the ℓ^∞ – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\| = \max_{1\leq j\leq n} |x_j|$. Then the operator norm of T is given by

$$\|T\| = \max_{1\leq i\leq m} \sum_{j=1}^n |T_{ij}|.$$

Exercise 2.7. Assume the norms on X and Y are the ℓ^2 – norms, i.e. for $x \in \mathbb{R}^n$, $\|x\|^2 = \sum_{j=1}^n x_j^2$. Show $\|T\|^2$ is the largest eigenvalue of the matrix $T^{tr}T : \mathbb{R}^n \to \mathbb{R}^n$. **Hint:** Use the spectral theorem for symmetric real matrices.

Exercise 2.8. Suppose that X is a normed space and there exists an uncountable set A and $\{x_t\}_{t\in A} \subset X$ such that $\|x_t - x_s\| \geq 1$ for all $s \neq t$ with $s, t \in A$. Show X is not separable.

Exercise 2.9. Let $X = \mathcal{B}\left(L^2(\mathbb{R}, dx), L^2(\mathbb{R}, dx)\right)$ be the Banach space of bounded operators from $L^2(\mathbb{R}, dx)$ to itself. Show X is not separable by showing;

$$\|T_t - T_s\|_{op} \geq \sqrt{2} \geq 1 \text{ for } s \neq t$$

where for $t \in \mathbb{R}$, $T_t \in X$ is define by

$$(T_t f)(x) = f(x + t) \text{ for } x \in \mathbb{R} \text{ and } f \in L^2(\mathbb{R}, dx).$$

As usual,

$$\|T\|_{op} := \sup_{\|f\|_{L^2(dx)}=1} \|Tf\|_{L^2(dx)}$$

and

$$\|f\|^2_{L^2(dx)} = \int_{\mathbb{R}} |f(x)|^2 \, dx.$$

Exercise 2.10. Let $X = \mathbb{N}$ and for $p, q \in [1, \infty)$ let $\|\cdot\|_p$ denote the $\ell^p(\mathbb{N})$ – norm. Show $\|\cdot\|_p$ and $\|\cdot\|_q$ are inequivalent norms for $p \neq q$ by showing

$$\sup_{f \neq 0} \frac{\|f\|_p}{\|f\|_q} = \infty \text{ if } p < q.$$

Exercise 2.11. Let $d : C(\mathbb{R}) \times C(\mathbb{R}) \to [0, \infty)$ be defined by

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n},$$

where $\|f\|_n := \sup\{|f(x)| : |x| \leq n\} = \max\{|f(x)| : |x| \leq n\}$.

1. Show that d is a metric on $C(\mathbb{R})$.
2. Show that a sequence $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ converges to $f \in C(\mathbb{R})$ as $n \to \infty$ iff f_n converges to f uniformly on bounded subsets of $\mathbb{R}$.
3. Show that $(C(\mathbb{R}), d)$ is a complete metric space.

Exercise 2.12. Let $X = C([0, 1], \mathbb{R})$ and for $f \in X$, let

$$\|f\|_1 := \int_0^1 |f(t)| \, dt.$$

Show that $(X, \|\cdot\|_1)$ is normed space and show by example that this space is **not** complete. Hint: For the last assertion find a sequence of $\{f_n\}_{n=1}^{\infty} \subset X$ which is "trying" to converge to the function $f = 1_{[\frac{1}{2}, 1]} \notin X$.

Exercise 2.13. Suppose X is a Banach space and $\{f_n : n \in \mathbb{N}\}$ is a sequence in X such that $\lim_{n \to \infty} f_n = f \in X$. Show $s_N := \frac{1}{N} \sum_{n=1}^{N} f_n$ for $N \in \mathbb{N}$ is also a convergent sequence and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_n = \lim_{N \to \infty} s_N = f.$$