Also let $\mathcal{P}$ denote those functions $g \in C^\infty(\mathbb{R}^n)$ such that $g$ and all of its derivatives have at most polynomial growth, i.e. $g \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{P}$ if for all multi-indices $\alpha$, there exists $N_\alpha < \infty$ such

$$\sup (1 + |x|)^{-N_\alpha} |\partial^\alpha g(x)| < \infty.$$  

(Notice that any polynomial function on $\mathbb{R}^n$ is in $\mathcal{P}$.)

**Remark 5.46.** Since $C^\infty_c(\mathbb{R}^n) \subset S \subset L^2(\mathbb{R}^n)$, it follows that $S$ is dense in $L^2(\mathbb{R}^n)$.

**Exercise 5.29.** Let

$$L = \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha$$

with $a_\alpha \in \mathcal{P}$. Show $L(S) \subset S$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $S$ for all multi-indices $\alpha$.

The quantum harmonic oscillator Hamiltonian on $L^2(\mathbb{R})$ is the unbounded operator,

$$H = -\frac{1}{2} \partial_x^2 + \frac{1}{2} M x^2$$

acting on $S$ — say.

**Definition 5.47 (Annihilation and Creation operators).** For $\alpha > 0$, let $a$ be the annihilation operator acting on $L^2(\mathbb{R})$ defined so that $D(a) = S$ and

$$(af)(x) := \sqrt{\frac{1}{2}} (xf(x) + \partial_x f(x)) \quad \text{for } f \in S.$$  

The corresponding creation operator is $a^\dagger$ — the formal adjoint of $a$, i.e.

$$(a^\dagger f)(x) := \sqrt{\frac{1}{2}} (xf(x) - \partial_x f(x)) \quad \text{for } f \in S.$$  

We further let

$$N := a^\dagger a = \frac{1}{2} (x - \partial_x)(x + \partial_x) = -\frac{1}{2} \partial_x^2 + \frac{1}{2} x^2 - \frac{1}{2} I$$

so that $H = N + \frac{1}{2} I$.

Notice that both the creation $(a^\dagger)$ and annihilation $(a)$ operators preserve $S$ and satisfy the canonical commutation relations (CCRs),

$$[a, a^\dagger] = I|_S.$$  

Let

$$\Omega_0(x) := \frac{1}{\sqrt{4\pi}} \exp \left(-\frac{1}{2} x^2\right) \quad \text{and} \quad \Omega_n := \frac{1}{\sqrt{n!}} a^\dagger n \Omega_0.$$  

**Lemma 5.48.** The function $\{\Omega_n\}_{n=0}^\infty$ are total in $L^2(\mathbb{R}, m)$.

**Proof.** Let us observe that there exists $c_n > 0$ so that

$$\Omega_n(x) = [c_n x^n + \ldots] \Omega_0(x)$$

as is easily proved by induction using $-\partial_x \Omega_0(x) = x \Omega_0(x)$. Therefore the span $\{\Omega_n\}_{n=0}^\infty$ consists of all function of the form, $p(x) \Omega_0(x)$, where $p \in \mathbb{C}[x]$. If $g \in L^2(m)$, then $G := g/\Omega_0$ is $L^2(\Omega_0^2 dm)$ and polynomials are dense in $L^2(\Omega_0^2 dm)$. Thus given $\varepsilon > 0$, there $p \in \mathbb{C}[x]$ so that

$$\varepsilon > \int_{\mathbb{R}} \left| \frac{g}{\Omega_0}(x) - p(x) \right|^2 \Omega_0^2(x) \, dx = \int_{\mathbb{R}} \left| g(x) - p(x) \Omega_0(x) \right|^2 \, dx.$$  

The following theorem summarizes the basic well known and easily verified properties of these functions which essentially are all easy consequences of the commutation relations, $[a, a^\dagger] = I$ on $S$. We will provide a short proof of these well known results for the readers convenience.

**Theorem 5.49.** The functions $\{\Omega_n\}_{n=0}^\infty \subset S$ form a complete orthonormal basis for $L^2(m)$ which satisfy for all $n \in \mathbb{N}$,

$$a \Omega_n = \sqrt{n+1} \Omega_{n+1},$$

$$a^\dagger \Omega_n = \sqrt{n} \Omega_{n-1}$$

and

$$N \Omega_n = n \Omega_n$$

where $\Omega_{-1} \equiv 0$. 

**Convention:** $\Omega_n \equiv 0$ for all $n \in \mathbb{Z}$ with $n < 0$. 