Also let $\mathcal{P}$ denote those functions $g \in C^\infty(\mathbb{R}^n)$ such that g and all of its derivatives have at most polynomial growth, i.e. $g \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{P}$ iff for all multi-indices α, there exists $N_\alpha < \infty$ such

$$\sup (1+|x|)^{-N_\alpha} |\partial^\alpha g(x)| < \infty.$$

(Notice that any polynomial function on $\mathbb{R}^n$ is in $\mathcal{P}$.)

Remark 5.46. Since $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S} \subset L^2(\mathbb{R}^n)$, it follows that $\mathcal{S}$ is dense in $L^2(\mathbb{R}^n)$.

Exercise 5.29. Let

$$L = \sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha \tag{5.35}$$

with $a_\alpha \in \mathcal{P}$. Show $L(\mathcal{S}) \subset \mathcal{S}$ and in particular $\partial^\alpha f$ and $x^\alpha f$ are back in $\mathcal{S}$ for all multi-indices α.

The quantum harmonic oscillator Hamiltonian on $L^2(\mathbb{R})$ is the unbounded operator,

$$H = -\frac{1}{2}\partial_x^2 + \frac{1}{2} M_{x^2}$$

actin on $\mathcal{S}$ – say.

Definition 5.47 (Annihilation and Creation operators). *For > 0, let a be the **annihilation operator** acting on $L^2(m)$ defined so that $D(a) = \mathcal{S}$ and*

$$(af)(x) := \sqrt{\frac{1}{2}} (xf(x) + \partial_x f(x)) \text{ for } f \in \mathcal{S}. \tag{5.36}$$

*The corresponding **creation operator** is $a^\dagger$ – the formal adjoint of a, i.e.*

$$\left(a^\dagger f\right)(x) := \sqrt{\frac{1}{2}} (xf(x) - \partial_x f(x)) \text{ for } f \in \mathcal{S}. \tag{5.37}$$

We further let

$$N := a^\dagger a = \frac{1}{2}(x - \partial_x)(x + \partial_x) = -\frac{1}{2}\partial_x^2 + \frac{1}{2}x^2 - \frac{1}{2}I \tag{5.38}$$

so that $H = N + \frac{1}{2}I$.

Notice that both the creation $\left(a^\dagger\right)$ and annihilation (a) operators preserve $\mathcal{S}$ and satisfy the canonical commutation relations (CCRs),

$$\left[a, a^\dagger\right] = I|_{\mathcal{S}}. \tag{5.39}$$

Let

$$\Omega_0(x) := \frac{1}{\sqrt[4]{4\pi}} \exp\left(-\frac{1}{2}x^2\right) \text{ and } \left\{\Omega_n := \frac{1}{\sqrt{n!}} a^{\dagger n} \Omega_0\right\}_{n=0}^\infty. \tag{5.40}$$

Convention: $\Omega_n \equiv 0$ for all $n \in \mathbb{Z}$ with $n < 0$.

Lemma 5.48. *The function $\{\Omega_n\}_{n=0}^\infty$ are total in $L^2(\mathbb{R}, m)$.*

Proof. Let us observe that there exists $c_n > 0$ so that

$$\Omega_n(x) = [c_n x^n + \dots] \Omega_0(x)$$

as is easily proved by induction using $-\partial_x \Omega_0(x) = x\Omega_0(x)$. Therefore the span $\{\Omega_n\}_{n=0}^\infty$ consists of all function of the form, $p(x)\Omega_0(x)$, where $p \in \mathbb{C}[x]$. If $g \in L^2(m)$, then $G := g/\Omega_0$ is $L^2\left(\Omega_0^2 dm\right)$ and polynomials are dense in $L^2\left(\Omega_0^2 dm\right)$. Thus given $\varepsilon > 0$, there $p \in \mathbb{C}[x]$ so that

$$\varepsilon > \int_{\mathbb{R}} \left| \frac{g(x)}{\Omega_0(x)} - p(x) \right|^2 \Omega_0^2(x)\, dx = \int_{\mathbb{R}} |g(x) - p(x)\Omega_0(x)|^2 dx.$$

■

The following theorem summarizes the basic well known and easily verified properties of these functions which essentially are all easy consequences of the commutation relations, $\left[a, a^\dagger\right] = I$ on $\mathcal{S}$. We will provide a short proof of these well known results for the readers convenience.

Theorem 5.49. *The functions $\{\Omega_n\}_{n=0}^\infty \subset \mathcal{S}$ form an orthonormal basis for $L^2(m)$ which satisfy for all $n \in \mathbb{N}_0$,*

$$a\Omega_n = \sqrt{n}\Omega_{n-1}, \tag{5.41}$$

$$a^\dagger \Omega_n = \sqrt{n+1}\Omega_{n+1} \text{ and} \tag{5.42}$$

$$N\Omega_n = n\Omega_n \tag{5.43}$$

where $\Omega_{-1} \equiv 0$.