Part I

Rough Path Analysis

Here are a few suggested references for this course, [3, 6, 1]. The latter two references are downloadable if you are logging into MathSci net through your UCSD account. For a proof that all p – variation paths have some extension to a rough path see, [5] and also see [2, Theorem 9.12 and Remark 9.13].

From Feynman Heuristics to Brownian Motion

In the physics literature one often finds the following informal expression,

$$d\mu_T(\omega) = \text{“}\frac{1}{Z(T)} e^{-\frac{1}{2}\int_0^T |\omega'(\tau)|^2 d\tau} \mathcal{D}_T\omega\text{” for } \omega \in W_T, \tag{1.1}$$

where W_T is the set of continuous paths, $\omega : [0,T] \to \mathbb{R}$ (or $\mathbb{R}^d$), such that $\omega(0) = 0$,

$$\mathcal{D}_T\omega = \text{“}\prod_{0<t\le T} m(d\omega(t))\text{” } (m \text{ is Lebesgue measure here})$$

and $Z(T)$ is a normalization constant such that $\mu_T(W_T) = 1$.

We begin by giving meaning to this expression. For $0 \le s \le t \le T$, let

$$E_{[s,t]}(\omega) := \int_s^t |\omega'(\tau)|^2 d\tau.$$

If we decompose $\omega(\tau)$ as $\sigma(\tau) + \gamma(\tau)$ where

$$\sigma(\tau) := \omega(s) + \frac{\tau - s}{t - s}(\omega(t) - \omega(s)) \text{ and } \gamma(\tau) := \omega(\tau) - \sigma(\tau),$$

then we have, $\sigma\prime(\tau) = \frac{\omega(t)-\omega(s)}{t-s}$, $\gamma(s) = \gamma(t) = 0$, and hence

$$\begin{aligned}\int_s^t \sigma'(\tau)\cdot\gamma'(\tau)\,d\tau &= \int_s^t \sigma'(\tau)\cdot\gamma'(\tau)\,d\tau \\ &= \frac{\omega(t) - \omega(s)}{t-s}\cdot(\gamma(t) - \gamma(s)) = 0.\end{aligned}$$

Thus it follows that

$$\begin{aligned}E_{[s,t]}(\omega) &= E_{[s,t]}(\sigma) + E_{[s,t]}(\gamma) = \left|\frac{\omega(t) - \omega(s)}{t-s}\right|^2 (t-s) + E_{[s,t]}(\gamma) \\ &= \frac{|\omega(t) - \omega(s)|^2}{t-s} + E_{[s,t]}(\gamma).\end{aligned} \tag{1.2}$$

Thus if $f(\omega) = F\left(\omega|_{[0,s]}, \omega(t)\right)$, we will have,

$$\begin{aligned}&\frac{1}{Z_t}\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}E_t(\omega)}\mathcal{D}_t\omega \\ &\qquad = \frac{1}{Z_t}\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega) + E_{[s,t]}(\omega)\right]}\mathcal{D}_t\omega\end{aligned}$$

and now fixing $\omega|_{[0,s]}$ and $\omega(t)$ and then doing the integral over $\omega|_{(s,t)}$ implies,

$$\begin{aligned}&\int F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega) + E_{[s,t]}(\omega)\right]}\mathcal{D}_{(s,t)}\omega \\ &= \int F\left(\omega|_{[0,s]}, \omega(t)\right) e^{-\frac{1}{2}\left[E_s(\omega) + \frac{|\omega(t)-\omega(s)|^2}{t-s} + E_{[s,t]}(\gamma)\right]}\mathcal{D}_{(s,t)}\gamma \\ &= C(s,t)\int F\left(\omega|_{[0,s]}, \omega(t)\right)\frac{e^{-\frac{1}{2}E_s(\omega)}}{Z(s)} e^{-\frac{1}{2}\frac{|\omega(t)-\omega(s)|^2}{t-s}}.\end{aligned}$$

Multiplying this equation by $\frac{1}{Z_t}\mathcal{D}\omega_{[0,s]}\cdot d\omega(t)$ and integrating the result then implies,

$$\begin{aligned}&\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) d\mu_t(\omega) \\ &\qquad = \frac{C(s,t)}{Z_t}\int\left[\int_{\mathbb{R}^d} F\left(\omega|_{[0,s]}, y\right) e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]\frac{e^{-\frac{1}{2}E_s(\omega)}}{Z(s)}\mathcal{D}\omega_{[0,s]} \\ &\qquad = \frac{C(s,t)}{Z_t}\int_{W_s}\left[\int_{\mathbb{R}^d} F(\omega, y)\, e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]d\mu_s(\omega).\end{aligned}$$

Taking $F \equiv 1$ in this equation then implies,

$$\begin{aligned}1 &= \frac{C(s,t)}{Z_t}\int_{W_s}\left[\int_{\mathbb{R}^d} e^{-\frac{1}{2}\frac{|y-\omega(s)|^2}{t-s}}dy\right]d\mu_s(\omega) \\ &= \frac{C(s,t)}{Z_t}\int_{W_s}\left[(2\pi(t-s))^{d/2}\right]d\mu_s(\omega) = \frac{C(s,t)}{Z_t}(2\pi(t-s))^{d/2}.\end{aligned}$$

Thus the heuristic expression in Eq. (1.1) leads to the following **Markov property** for μ_t, namely.

Proposition 1.1 (Heuristic). *Suppose that $F : W_s \times \mathbb{R}^d \to \mathbb{R}$ is a reasonable function, then for any $t \ge s$ we have*

$$\int_{W_t} F\left(\omega|_{[0,s]}, \omega(t)\right) d\mu_t(\omega) = \int_{W_s}\left[\int_{\mathbb{R}^d} F(\omega, y)\, p_{t-s}(\omega(s), y)\, dy\right] d\mu_s(\omega),$$

where

$$p_s(x,y) := \left(\frac{1}{2\pi(t-s)}\right)^{d/2} e^{-\frac{1}{2}\frac{|y-x|^2}{t-s}}. \tag{1.3}$$

Corollary 1.2 (Heuristic). *If $0 = s_0 < s_1 < s_2 < \cdots < s_n = T$ and $f : \left(\mathbb{R}^d\right)^n \to \mathbb{R}$ is a reasonable function, then*

$$\int_{W_T} f(\omega(s_1), \dots, \omega(s_n))\, d\mu_T(\omega) = \int_{(\mathbb{R}^d)^n} f(y_1, \dots, y_n) \prod_{i=1}^n \left(p_{s_i - s_{i-1}}(y_{i-1}, y_i)\, dy_i\right) \tag{1.4}$$

where by convention, $y_0 = 0$.

Theorem 1.3 (Wiener 1923). *For all $t > 0$ there exists a unique probability measure, μ_t, on W_t, such that Eq. (1.4) holds for all n and all bounded measurable $f : \left(\mathbb{R}^d\right)^n \to \mathbb{R}$.*

Definition 1.4. *Let $B_t(\omega) := \omega(t)$. Then $\{B_t\}_{0 \le t \le T}$ as a process on (W_T, μ_T) is called **Brownian motion.** We further write $\mathbb{E} f$ for $\int_{W_T} f(\omega)\, d\mu_T(\omega)$.*

The following lemma is useful for computational purposes involving Brownian motion and follows readily form from Eq. (1.4).

Lemma 1.5. *Suppose that $0 = s_0 < s_1 < s_2 < \cdots < s_n = t$ and $f_i : \mathbb{R}^d \to \mathbb{R}$ are reasonable functions, then*

$$\mathbb{E}\left[\prod_{i=1}^n f_i\left(B_{s_i} - B_{s_{i-1}}\right)\right] = \prod_{i=1}^n \mathbb{E}\left[f_i\left(B_{s_i} - B_{s_{i-1}}\right)\right], \tag{1.5}$$

$$\mathbb{E}\left[f(B_t - B_s)\right] = \mathbb{E}\left[f(B_{t-s})\right], \tag{1.6}$$

and

$$\mathbb{E}\left[f(B_t)\right] = \mathbb{E} f\left(\sqrt{t} B_1\right). \tag{1.7}$$

As an example let us observe that

$$\mathbb{E} B_t = \int y p_t(y)\, dy = 0,$$

$$\mathbb{E} B_t^2 = t \mathbb{E} B_1^2 = t \int y^2 p_1(y)\, dy = t \cdot 1,$$

and for $s < t$,

$$\mathbb{E}\left[B_t B_s\right] = \mathbb{E}\left[(B_t - B_s) B_s\right] + \mathbb{E} B_s^2 = \mathbb{E}(B_t - B_s) \cdot \mathbb{E} B_s + s = s$$

and

$$\mathbb{E}\left[|B_t - B_s|^p\right] = |t-s|^{p/2}\, \mathbb{E}\left[|B_1|^p\right] = C_p |t-s|^{p/2}. \tag{1.8}$$

1.1 Construction and basic properties of Brownian motion

In this section we sketch one method of constructing Wiener measure ore equivalently Brownian motion. We begin with the existence of a measure ν_T on the $\tilde{W}_T := \prod_{0 \le s \le T} \bar{\mathbb{R}}$ which satisfies Eq. (1.4) where $\bar{\mathbb{R}}$ is a compactification of $\mathbb{R}$ – for example either one point compactificatoin so that $\bar{\mathbb{R}} \cong S^1$.

Theorem 1.6 (Kolmogorov's Existence Theorem). *There exists a probability measure, ν_T, on $\tilde{W}_T$ such that Eq. (1.4) holds.*

Proof. For a function $F(\omega) := f(\omega(s_1), \dots, \omega(s_n))$ where $f \in C\left(\bar{\mathbb{R}}^n, \mathbb{R}\right)$, define

$$I(F) := \int_{\mathbb{R}^n} f(y_1, \dots, y_n) \prod_{i=1}^n \left(p_{s_i - s_{i-1}}(y_{i-1}, y_i)\, dy_i\right).$$

Using the semi-group property;

$$\int_{\mathbb{R}^d} p_t(x,y)\, p_s(y,z)\, dy = p_{s+t}(x,z)$$

along with the fact that $\int_{\mathbb{R}^d} p_t(x,y)\, dy = 1$ for all $t > 0$, one shows that $I(F)$ is well defined independently of how we represent F as a "finitely based" continuous function.

By Tychonoff's Theorem $\tilde{W}_T$ is a compact Hausdorff space. By the Stone Weierstrass Theorem, the finitely based continuous functions are dense inside of $C\left(\tilde{W}_T\right)$. Since $|I(F)| \le \|F\|_\infty$ for all finitely based continuous functions, we may extend I uniquely to a positive continuous linear functional on $C\left(\tilde{W}_T\right)$. An application of the Riesz Markov theorem now gives the existence of the desired measure, ν_T. ∎

Theorem 1.7 (Kolmogorov's Continuity Criteria). *Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space and $\tilde{X}_t : \Omega \to S$ is a process for $t \in [0,T]$ where (S, ρ) is a complete metric space. Assume there exists positive constants, ε, β, and C, such that*

$$\mathbb{E}[\rho(\tilde{X}_t, \tilde{X}_s)^\varepsilon] \le C |t-s|^{1+\beta} \tag{1.9}$$

for all $s, t \in [0,T]$ *. Then for any* $\alpha \in (0, \beta/\varepsilon)$ *there is a modification,* X*, of* $\tilde{X}$ *(i.e.* $P\left(X_t = \tilde{X}_t\right) = 1$ *for all* t*) which is* α*–Hölder continuous. Moreover, there is a random variable* K_α *such that,*

$$\rho(X_t, X_s) \leq K_\alpha |t-s|^\alpha \text{ for all } s, t \in [0,T] \tag{1.10}$$

and $\mathbb{E}K_\alpha^p < \infty$ *for all* $p < \frac{\beta - \alpha\varepsilon}{1-\alpha}$*. (For the proof of this theorem see Section* **??** *below.)*

Corollary 1.8. *Let* $\tilde{B}_t : \tilde{W}_T \to \mathbb{R}$ *be the projection map,* $\tilde{B}_t(\omega) = \omega(t)$ *. Then there is a modifications,* $\{B_t\}$ *of* $\left\{\tilde{B}_t\right\}$ *for which* $t \to B_t$ *is* α *– Hölder continuous* ν_T *– almost surely for any* $\alpha \in (0, 1/2)$ *.*

Proof. Applying Theorem 1.7 with $\varepsilon := p$ and $\beta := p/2 - 1$ for any $p \in (2, \infty)$ shows there is a modification $\{B_t\}_{t \geq 0}$ of $\left\{\tilde{B}_t\right\}$ which is almost surely α – Hölder continuous for any

$$\alpha \in (0, \beta/\varepsilon) = \left(0, \frac{p/2 - 1}{p}\right) = (0, 1/2 - 1/p).$$

Letting $p \to \infty$ shows that $\{B_t\}_{t \geq 0}$ is almost surely α – Hölder continuous for all $\alpha < 1/2$. ■

We will see shortly that these Brownian paths are very rough. Before we do this we will pause to develope a quatitative measurement of roughness of a continuous path.

p – Variations and Controls

Let (E,d) be a metric space which will usually be assumed to be complete.

Definition 2.1. *Let* $0 \le a < b < \infty$. *Given a* ***partition*** $\Pi := \{a = t_0 < t_1 < \cdots < t_n = b\}$ *of* $[a,b]$ *and a function* $Z \in C([a,b],E)$, *let* $(t_i)_- := t_{i-1}$, $(t_i)_+ := t_{i+1}$, *with the convention that* $t_{-1} := t_0 = a$ *and* $t_{n+1} := t_n = T$. *Furthermore for* $1 \le p < \infty$ *let*

$$V_p(Z:\Pi) := \left(\sum_{j=1}^n d^p\left(Z_{t_j}, Z_{t_{j-1}}\right)\right)^{1/p} = \left(\sum_{t\in\Pi} d^p\left(Z_t, Z_{t_-}\right)\right)^{1/p}. \tag{2.1}$$

Furthermore, let $\mathcal{P}(a,b)$ *denote the collection of partitions of* $[a,b]$. *Also let* $\operatorname{mesh}(\Pi) := \max_{t\in\Pi}|t-t_-|$ *be the* ***mesh*** *of the partition,* Π.

Definition 2.2. *and* $Z \in C([a,b],E)$. *For* $1 \le p < \infty$, *the* ***p - variation*** *of* Z *is;*

$$V_p(Z) := \sup_{\Pi\in\mathcal{P}(a,b)} V_p(Z:\Pi) = \sup_{\Pi\in\mathcal{P}(a,b)} \left(\sum_{j=1}^n d^p\left(Z_{t_j}, Z_{t_{j-1}}\right)\right)^{1/p}. \tag{2.2}$$

Moreover if $Z \in C([0,T],E)$ *and* $0 \le a \le b \le T$, *we let*

$$\omega_{Z,p}(a,b) := \left[\nu_p\left(Z|_{[a,b]}\right)\right]^p = \sup_{\Pi\in\mathcal{P}(a,b)} \sum_{j=1}^n d^p\left(Z_{t_j}, Z_{t_{j-1}}\right). \tag{2.3}$$

Remark 2.3. We can define $V_p(Z)$ for $p \in (0,1)$ as well but this is not so interesting. Indeed if $0 \le s \le T$ and $\Pi \in \mathcal{P}(0,T)$ is a partition such that $s \in \Pi$, then

$$\begin{aligned} d(Z(s),Z(0)) &\le \sum_{t\in\Pi} d(Z(t),Z(t_-)) = \sum_{t\in\Pi} d^{1-p}(Z(t),Z(t_-))\, d^p(Z(t),Z(t_-)) \\ &\le \max_{t\in\Pi} d^{1-p}(Z(t),Z(t_-)) \cdot V_p^p(Z:\Pi) \\ &\le \max_{t\in\Pi} d^{1-p}(Z(t),Z(t_-)) \cdot V_p^p(Z). \end{aligned}$$

Using the uniform continuity of Z (or $d(Z(s),Z(t))$ if you wish) we know that $\lim_{|\Pi|\to 0}\max_{t\in\Pi} d^{1-p}(Z(t),Z(t_-)) = 0$ and hence that,

$$d(Z(s),Z(0)) \le \lim_{|\Pi|\to 0}\max_{t\in\Pi} d^{1-p}(Z(t),Z(t_-)) \cdot V_p^p(Z) = 0.$$

Thus we may conclude $Z(s) = Z(0)$, i.e. Z must be constant.

Lemma 2.4. *Let* $\{a_i > 0\}_{i=1}^n$, *then*

$$\left(\sum_{i=1}^n a_i^p\right)^{1/p} \text{ is decreasing in } p \text{ and}$$

$$\varphi(p) := \ln\left(\sum_{i=1}^n a_i^p\right) \text{ is convex in } p.$$

Proof. Let $f(i) = a_i$ and $\mu(\{i\}) = 1$ be counting measure so that

$$\sum_{i=1}^n a_i^p = \mu(f^p) \text{ and } \varphi(p) = \ln\mu(f^p).$$

Using $\frac{d}{dp}f^p = f^p \ln f$, it follows that and

$$\begin{aligned} \varphi'(p) &= \frac{\mu(f^p\ln f)}{\mu(f^p)} \text{ and} \\ \varphi''(p) &= \frac{\mu\left(f^p\ln^2 f\right)}{\mu(f^p)} - \left[\frac{\mu(f^p\ln f)}{\mu(f^p)}\right]^2. \end{aligned}$$

Thus if we let $\mathbb{E}X := \mu(f^pX)/\mu(f^p)$, we have shown, $\varphi'(p) = \mathbb{E}[\ln f]$ and

$$\varphi''(p) = \mathbb{E}\left[\ln^2 f\right] - (\mathbb{E}[\ln f])^2 = \operatorname{Var}(\ln f) \ge 0$$

which shows that φ is convex in p.

Now let us shows that $\|f\|_p$ is decreasing in in p. To this end we compute,

$$\begin{aligned}\frac{d}{dp}\left[\ln\|f\|_p\right] &= \frac{d}{dp}\left[\frac{1}{p}\varphi(p)\right] = \frac{1}{p}\varphi'(p) - \frac{1}{p^2}\varphi(p)\\ &= \frac{1}{p^2\mu(f^p)}\left[p\mu(f^p\ln f) - \mu(f^p)\ln\mu(f^p)\right]\\ &= \frac{1}{p^2\mu(f^p)}\left[\mu(f^p\ln f^p) - \mu(f^p)\ln\mu(f^p)\right]\\ &= \frac{1}{p^2\mu(f^p)}\left[\mu\left(f^p\ln\frac{f^p}{\mu(f^p)}\right)\right].\end{aligned}$$

Up to now our computation has been fairly general. The point where μ being counting measure comes in is that in this case $\mu(f^p) \geq f^p$ everywhere and therefore $\ln\frac{f^p}{\mu(f^p)} \leq 0$ and therefore, $\frac{d}{dp}\left[\ln\|f\|_p\right] \leq 0$ as desired.

Alternative proof that $\|f\|_p$ is decreasing in p. If we let $q = p + r$, then

$$\|a\|_q^q = \sum_{j=1}^n a_j^{p+r} \leq \left(\max_j a_j\right)^r \cdot \sum_{j=1}^n a_j^p \leq \|a\|_p^r \cdot \|a\|_p^p = \|a\|_p^q,$$

wherein we have used,

$$\max_j a_j = \left(\max_j a_j^p\right)^{1/p} \leq \left(\sum_{j=1}^n a_j^p\right)^{1/p} = \|a\|_p.$$

■

Remark 2.5. It is not too hard to see that the convexity of φ is equivalent to the interpolation inequality,

$$\|f\|_{p_s} \leq \|f\|_{p_0}^{1-s} \cdot \|f\|_{p_1}^s,$$

where $0 \leq s \leq 1$, $1 \leq p_0, p_1$, and

$$\frac{1}{p_s} := (1-s)\frac{1}{p_0} + s\frac{1}{p_1}.$$

This interpolation inequality may be proved via Hölder's inequality.

Corollary 2.6. *The function $V_p(Z)$ is a decreasing function of p and $\ln V_p(Z)^p$ is a convex function of p where they are finite. Moreover, for all $p_0 > 1$,*

$$\lim_{p\downarrow p_0} V_p(Z) = V_{p_0}(Z). \tag{2.4}$$

and $p \to V_p(Z)$ is continuous on the set of p's where $V_p(Z)$ is finite.

Proof. Given Lemma 2.4, it suffices to prove Eq. (2.4) and the continuity assertion on $p \to V_p(Z)$. Since $p \to V_p(Z)$ is a decreasing function, we know that $\lim_{p\uparrow p_0} V_p(Z)$ and $\lim_{p\downarrow p_0} V_p(Z)$ always exists and also that $\lim_{p\downarrow p_0} V_p(Z) = \sup_{p>p_0}\sup_{\Pi} V_p(Z:\Pi)$. Therefore,

$$\lim_{p\downarrow p_0} V_p(Z) = \sup_{p>p_0}\sup_{\Pi} V_p(Z:\Pi) = \sup_{\Pi}\sup_{p>p_0} V_p(Z:\Pi) = \sup_{\Pi} V_{p_0}(Z:\Pi) = V_{p_0}(Z)$$

which proves Eq. (2.4). The continuity of $V_p(Z) = \exp\left(\frac{1}{p}\ln V_p(Z)^p\right)$ follows directly from the fact that $\ln V_p(Z)^p$ is convex in p and that convex functions are continuous (where finite).

Here is a proof for this case. Let $\varphi(p) := \ln V_p(Z)^p$, $1 \leq p_0 < p_1$ such that $V_{p_0}(Z) < \infty$, and $p_s := (1-s)p_0 + sp_1$, then

$$\varphi(p_s) \leq (1-s)\varphi(p_0) + s\varphi(p_1).$$

Letting $s \uparrow 1$ then implies $p_s \uparrow p_1$ and $\varphi(p_1-) \leq \varphi(p_1)$, i.e. $V_{p_1-} \leq V_{p_1} \leq V_{p_1-}$. Therefore $V_{p_1-} = V_{p_1}$ and along with Eq. (2.4) proves the continuity of $p \to V_p(Z)$. ■

2.1 Computing $V_p(x)$

How do we actually compute $V_p(x) := V_p(x; 0, T)$ for a given path $x \in C([0,T], \mathbb{R})$, even a very simple one? Suppose x is piecewise linear, with corners at the points $0 = s_0, s_1, \ldots, s_m = T$. Intuitively it would seem that the p-variation should be given by choosing the corners to be the partition points. That is, if $S = \{s_0, \ldots, s_n\}$ is the partition of corner points, we might think that $V_p(x) = V_p(x; S)$. Well, first we would have to leave out any corner which is not a local extremum (because of Lemma 2.8 below). But even then, this is not generally true as is seen in Example 2.9 below.

Lemma 2.7. *For all $a, b \geq 0$ and $p \geq 1$,*

$$(a+b)^p \geq a^p + b^p \tag{2.5}$$

and the inequality is strict if $a, b > 0$ and $p > 1$.

Proof. Observe that $(a+b)^p \geq a^p + b^p$ happens iff

$$1 \geq \left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p$$

which obviously holds since

$$\left(\frac{a}{a+b}\right)^p + \left(\frac{b}{a+b}\right)^p \leq \frac{a}{a+b} + \frac{b}{a+b} = 1.$$

Moreover the latter inequality is strict if if $a, b > 0$ and $p > 1$. ■

Lemma 2.8. *Let x be a path, and $D = \{t_0, \ldots, t_n\}$ be a partition. Suppose x is monotone increasing (decreasing) on $[t_{i-1}, t_{i+1}]$. Then if $D' = D \backslash \{t_i\}$, $V_p(x : D\prime) \geq V_p(x : D)$. If x is strictly increasing and $p > 1$, the inequality is strict.*

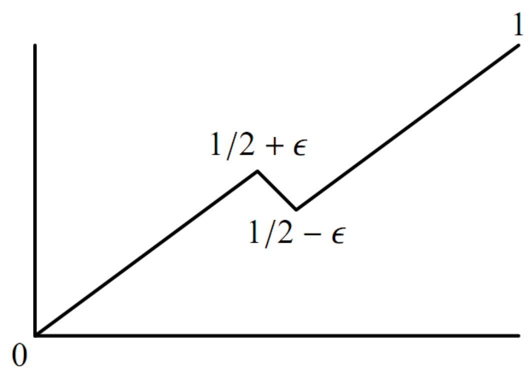
Proof. From Eq. (2.5) it follows

$$\begin{aligned} V_p(x : D\prime)^p - V_p(x : D)^p &= (x(t_{i+1}) - x(t_{i-1}))^p - (x(t_{i+1}) - x(t_i))^p - (x(t_i) - x(t_{i-1}))^p \\ &= \left(\Delta_{t_i} x + \Delta_{t_{i+1}} x\right)^p - (\Delta_{t_i} x)^p - \left(\Delta_{t_{i+1}} x\right)^p \geq 0 \end{aligned}$$

and the inequality is strict if $\Delta_{t_i} x > 0$, $\Delta_{t_{i+1}} x > 0$ and $p > 1$. ∎

In other words, on any monotone increasing segment, we should not include any intermediate points, because they can only hurt us.

Example 2.9. Consider a path like the following: If we partition $[0, T]$ at the



corner points, then

$$V_p(x : S)^p = (\frac{1}{2} + \epsilon)^p + (2\epsilon)^p + (\frac{1}{2} + \epsilon)^p \approx 2(\frac{1}{2})^p < 1$$

by taking ϵ small. On the other hand, taking the trivial partition $D = \{0, T\}$, $V_p(x : D) = 1$, so $V_p(x : S) < 1 \leq V_p(x)$ and in this case using all of local minimum and maximum does not maximize the $p-$variation.

The clean proof of the following theorem is due to Thomas Laetsch.

Theorem 2.10. *If $x : [0, T] \to \mathbb{R}$ having only finitely many local extremum in $(0, T)$ located at $\{s_1 < \cdots < s_{n-1}\}$. Then*

$$V_p(x) = \sup\{V_p(x : D) : \{0, T\} \subset D \subset S\},$$

where $S = \{0 = s_0 < s_1 < \cdots < s_n = T\}$.

Proof. Let $D = \{0 = t_0 < t_1 < \cdots < t_r = T\} \in \mathcal{P}(0, T)$ be an arbitrary partition of $[0, T]$. We are going to prove by induction that there is a partition $\Pi \subset S$ such that $V_p(x : D) \leq V_p(x : \Pi)$. The proof will be by induction on $n := \#(D \setminus S)$. If $n = 0$ there is nothing to prove. So let us now suppose that the theorem holds at some level $n \geq 0$ and suppose that $\#(D \setminus S) = n + 1$. Let $1 \leq k < r$ be chosen so that $t_k \in D \setminus S$. If $x(t_k)$ is between $x(t_{k-1})$ and $x(t_{k+1})$ (i.e. $(x(t_{k-1}), x(t_k), x(t_{k+1}))$ is a monotonic triple), then according Lemma 2.8 we will have $V_p(x : D) \leq V_p(x : D \setminus \{t_k\})$ and since $\#[(D \setminus \{t_k\}) \setminus S] = n$, the induction hypothesis implies there exists a partition, $\Pi \subset S$ such that

$$V_p(x : D) \leq V_p(x : D \setminus \{t_k\}) \leq V_p(x : \Pi).$$

Hence we may now assume that either $x(t_k) < \min(x(t_{k-1}), x(t_{k+1}))$ or $x(t_k) > \max(x(t_{k-1}), x(t_{k+1}))$. In the first case we let $t_k^* \in (t_{k-1}, t_{k+1})$ be a point where $x|_{[t_{k-1}, t_{k+1}]}$ has a minimum and in the second let $t_k^* \in (t_{k-1}, t_{k+1})$ be a point where $x|_{[t_{k-1}, t_{k+1}]}$ has a maximum. In either case if $D^* := (D \setminus \{t_k\}) \cup \{t_k^*\}$ we will have $V_p(x : D) \leq V_p(x : D^*)$ and $\#(D^* \setminus S) = n$. So again the induction hypothesis implies there exists a partition $\Pi \subset S$ such that

$$V_p(x : D) \leq V_p(x : D^*) \leq V_p(x : \Pi).$$

From these considerations it follows that

$$V_p(x : D) \leq \sup\{V_p(x : \Pi) : \Pi \in \mathcal{P}(0, T) \text{ s.t. } \Pi \subset S\}$$

and therefore

$$\begin{aligned} V_p(x) &= \sup\{V_p(x : D) : D \in \mathcal{P}(0, T)\} \\ &\leq \sup\{V_p(x : \Pi) : \Pi \in \mathcal{P}(0, T) \text{ s.t. } \Pi \subset S\} \leq V_p(x). \end{aligned}$$

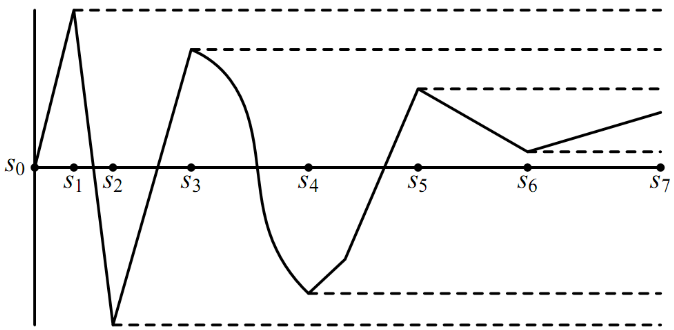
∎

Let us now suppose that x is (say) monotone increasing (not strictly) on $[s_0, s_1]$, monotone decreasing on $[s_1, s_2]$, and so on. Thus $s_0, s_2, \ldots$ are local minima, and $s_1, s_3, \ldots$ are local maxima. (If you want the reverse, just replace x with $-x$, which of course has the same p-variation.)

Definition 2.11. *Say that $s \in [0, T]$ is a **forward maximum** for x if $x(s) \geq x(t)$ for all $t \geq s$. Similarly, s is a **forward minimum** if $x(s) \leq x(t)$ for all $t \geq s$.*

Definition 2.12. *Suppose x is piecewise monotone, as above, with extrema $\{s_0, s_1, \ldots\}$. Suppose further that $s_2, s_4, \ldots$ are not only local minima but also forward minima, and that $s_1, s_3, \ldots$ are both local and forward maxima. Then we will say that x is **jog-free**.*

Note that $s_0 = 0$ does not *have to be a forward extremum. This is in order to admit a path with $x(0) = 0$ which can change signs.*



Here is an example.

Remark 2.13. Here is another way to state the jog-free condition. Let x be piecewise monotone with extrema $s_0, s_1, \ldots$. Let $\xi_i = |x(s_{i+1}) - x(s_i)|$. Then x is jog-free iff $\xi_1 \geq \xi_2 \geq \ldots$. The idea is that the oscillations are shrinking. (Notice that we don't need $\xi_0 \geq \xi_1$; this is because $s_0 = 0$ is not required to be a forward extremum.)

Remark 2.14. It is also okay if $s_1, s_2, \ldots$ are *backwards* extrema; this corresponds to the oscillations getting larger. Just reverse time, replacing $x(t)$ by $x(T-t)$, which again doesn't change the p-variation. Note that if ξ_i are as above, this corresponds to having $\xi_0 \leq \xi_1 \leq \xi_2 \leq \ldots$ (note that ξ_0 is included now, but ξ_{m-1} would not be). This case seems less useful, however.

Lemma 2.15. *Let x be jog-free with extrema $s_0, \ldots, s_m$. Let $D = \{t_0, \ldots, t_n\}$ be any partition not containing all the s_j. Then there is some $s_j \notin D$ such that if $D' = D \cup \{s_j\}$, $V_p(x : D\prime) \geq V_p(x : D)$.*

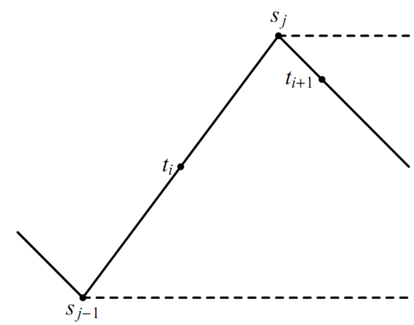
Proof. Let s_j be the first extremum not contained in D (note $s_0 = 0 \in D$ already, so j is at least 1 and s_j is also a forward extremum). Let t_i be the last element of D less than s_j. Note that $s_{j-1} \leq t_i < s_j < t_{i+1}$.

Now x is monotone on $[s_{j-1}, s_j]$; say WLOG it's monotone increasing, so that s_j is a local maximum and also a forward maximum. Since $t_i \in [s_{j-1}, s_j]$, where x is monotone increasing, $x(s_j) \geq x(t_i)$. And since s_j is a forward maximum, $x(s_j) \geq x(t_{i+1})$.

Therefore we have

$$x(s_j) - x(t_i) \geq x(t_{i+1}) - x(t_i)$$
$$x(s_j) - x(t_{i+1}) \geq x(t_i) - x(t_{i+1}).$$

One of the quantities on the right is equal to $|x(t_{i+1}) - x(t_i)|$, and so it follows that



$$|x(s_j) - x(t_i)|^p + |x(s_j) - x(t_{i+1})|^p \geq |x(t_{i+1}) - x(t_i)|^p$$

since one of the terms on the left is already $\geq$ the term on the right. This shows that $V_p(x : D\prime)^p \geq V_p(x : D)^p$. ∎

In other words, we should definitely include the extreme points, because they can only help.

Putting these together yields the desired result.

Proposition 2.16. *If x is jog-free with extrema $S = \{s_0, \ldots, s_m\}$, then $V_p(x) = V_p(x : S) = (\sum \xi_i^p)^{1/p}$.*

Proof. Fix $\epsilon > 0$, and let D be a partition such that $V_p(x : D) \geq V_p(x) - \epsilon$. By repeatedly applying Lemma 2.15, we can add the points of S to D one by one (in some order), and only increase the p-variation. So $V_p(x : D \cup S) \geq V_p(x : D)$. Now, if $t \in D \backslash S$, it is inside some interval $[s_j, s_{j+1}]$ on which x is monotone, and so by Lemma 2.8 t can be removed from $D \cup S$ to increase the p-variation. Removing all such points one by one (in any order), we find that $V_p(x : S) \geq V_p(x : D \cup S)$. Thus we have $V_p(x : S) \geq V_p(x : D) \geq V_p(x) - \epsilon$; since ϵ was arbitrary we are done. ∎

Notice that we only considered the case of jog-free paths with only finitely many extrema. Of course, in order to get infinite p-variation for any p we would need infinitely many extrema. Let's just check that the analogous result holds there.

Proposition 2.17. *Suppose we have a sequence $s_0, s_1, \ldots$ increasing to T, where x is alternately monotone increasing and decreasing on the intervals $[s_j, s_{j+1}]$. Suppose also that the s_j are forward extrema for x. Letting $\xi_j = |x(s_{j+1}) - x(s_j)|$ as before, we have*

$$V_p(x) = \left(\sum_{j=0}^{\infty} \xi_j^p \right)^{1/p}.$$

Actually, the extreme points s_j can converge to some earlier time than T, but x will have to be constant after that time.

Proof. For any m, we have $\sum_{j=0}^{m} \xi_j^p = V_p(x:D)^p$ for $D = \{s_0, \ldots, s_{m+1}\}$, so $V_p(x)^p \geq \sum_{j=0}^{m} \xi_j^p$. Passing to the limit, $V_p(x)^p \geq \sum_{j=0}^{\infty} \xi_j^p$.

For the reverse inequality, let $D = \{0 = t_0, t_1, \ldots, t_n = T\}$ be a partition with $V_p(x:D) \geq V_p(x) - \epsilon$. Choose m so large that $s_m > t_{n-1}$. Let $S = \{s_0, \ldots, s_m, T\}$, then by the same argument as in Proposition 2.16 we find that $V_p(x:S) \geq V_p(x:D)$. (Previously, the only way we used the assumption that S contained *all* extrema $s_j$was in order to have every $t_i \in D \backslash S$ contained in some monotone interval $[s_j, s_{j+1}]$. That is still the case here; we just take enough s_j's to ensure that we can surround each t_i. We do not need to surround $t_n = T$, since it is already in S.)

But $V_p(x:S)^p = \sum_{j=0}^{m-1} \xi_j^p \leq \sum_{j=0}^{\infty} \xi_j^p$, and so we have that

$$\left(\sum_{j=0}^{\infty} \xi_j^p\right)^{1/p} \geq V_p(x:D) \geq V_p(x) - \epsilon.$$

ϵ was arbitrary and we are done. ■

2.2 Brownian Motion in the Rough

Corollary 2.18. *For all $p > 2$ and $T < \infty$, $V_p\left(B|_{[0,T]}\right) < \infty$ a.s. (We will see later that $V_p\left(B|_{[0,T]}\right) = \infty$ a.s. for all $p < 2$.)*

Proof. By Corollary 1.8, there exists $K_p < \infty$ a.s. such that

$$|B_t - B_s| \leq K_p |t-s|^{1/p} \text{ for all } 0 \leq s, t \leq T. \tag{2.6}$$

Thus we have

$$\sum_i |\Delta_i B|^p \leq \sum_i \left(K_p |t_i - t_{i-1}|^{1/p}\right)^p \leq \sum_i K_p^p |t_i - t_{i-1}| = K_p^p T$$

and therefore, $V_p\left(B|_{[0,T]}\right) \leq K_p^p T < \infty$ a.s. ■

Proposition 2.19 (Quadratic Variation). *Let $\{\Pi_m\}_{m=1}^{\infty}$ be a sequence of partition of $[0,T]$ such that $\lim_{m\to\infty} |\Pi_m| = 0$ and define $Q_m := V_2^2(B:\Pi_m)$. Then*

$$\lim_{m\to\infty} \mathbb{E}\left[(Q_m - T)^2\right] = 0 \tag{2.7}$$

and if $\sum_{m=1}^{\infty} \text{mesh}(\Pi_m) < \infty$ then $\lim_{m\to\infty} Q_m = T$ a.s. This result is often abbreviated by the writing, $dB_t^2 = dt$.

Proof. Let N be an $N(0,1)$ random variable, $\Delta t := t - t_-$, $\Delta_t B := B_t - B_{t_-}$ and observe that $\Delta_t B \sim \sqrt{\Delta t} N$. Thus we have,

$$\mathbb{E} Q_m = \sum_{t\in\Pi_m} \mathbb{E}(\Delta_t B)^2 = \sum_{t\in\Pi_m} \Delta t = T.$$

Let us define

$$\begin{aligned} \text{Cov}(A,B) &:= \mathbb{E}[AB] - \mathbb{E}A \cdot \mathbb{E}B \text{ and} \\ \text{Var}(A) &:= \text{Cov}(A,A) = \mathbb{E}A^2 - (\mathbb{E}A)^2 = \mathbb{E}\left[(A - \mathbb{E}A)^2\right]. \end{aligned}$$

and observe that

$$\text{Var}\left(\sum_{i=1}^n A_i\right) = \sum_{i=1}^n \text{Var}(A_i) + \sum_{i\neq j} \text{Cov}(A_i, A_j).$$

As $\text{Cov}(\Delta_t B, \Delta_s B) = 0$ if $s \neq t$, we may use the above computation to conclude,

$$\begin{aligned} \text{Var}(Q_m) &= \sum_{t\in\Pi} \text{Var}((\Delta_t B)^2) = \sum_{t\in\Pi} \text{Var}(\Delta t \cdot N^2) \\ &= \text{Var}(N^2) \sum_{t\in\Pi} (\Delta t)^2 \leq \text{Var}(N^2) |\Pi_m| \sum_{t\in\Pi} \Delta t \\ &= T \cdot \text{Var}(N^2) |\Pi_m| \to 0 \text{ as } m \to \infty. \end{aligned}$$

(By explicit Gaussian integral computations,

$$\text{Var}(N^2) = \mathbb{E}N^4 - \left(\mathbb{E}N^2\right)^2 = 3 - 1 = 2 < \infty.)$$

Thus we have shown

$$\lim_{m\to\infty} \mathbb{E}\left[(Q_m - T)^2\right] = \lim_{m\to\infty} \mathbb{E}\left[(Q_m - \mathbb{E}Q)^2\right] = \lim_{m\to\infty} \text{Var}(Q_m) = 0.$$

If $\sum_{m=1}^{\infty} |\Pi_m| < \infty$, then

$$\begin{aligned} \mathbb{E}\left[\sum_{m=1}^{\infty} (Q_m - T)^2\right] &= \sum_{m=1}^{\infty} \mathbb{E}(Q_m - T)^2 = \sum_{m=1}^{\infty} \text{Var}(Q_m) \\ &\leq \text{Var}(N^2) \cdot T \cdot \sum_{m=1}^{\infty} \text{mesh}(\Pi_m) < \infty \end{aligned}$$

from which it follows that $\sum_{m=1}^{\infty} (Q_m - T)^2 < \infty$ a.s. In particular $(Q_m - T) \to 0$ almost surely. ■

Proposition 2.20. *If $p > q \geq 1$ and $V_q(Z) < \infty$, then $\lim_{|\Pi|\to 0} V_p(Z : \Pi) = 0$.*

Proof. Let $\Pi \in \mathcal{P}(0,T)$, then

$$
\begin{aligned}
V_p^p(Z : \Pi) &= \sum_{t\in\Pi} d^p(Z(t), Z(t_-)) = \sum_{t\in\Pi} d^{p-q}(Z(t), Z(t_-))\, d^q(Z(t), Z(t_-)) \\
&\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot \sum_{t\in\Pi} d^q(Z(t), Z(t_-)) \\
&\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z : \Pi) \\
&\leq \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z).
\end{aligned}
$$

Thus, by the uniform continuity of $Z|_{[0,T]}$ we have

$$\limsup_{|\Pi|\to 0} V_p(Z : \Pi) \leq \limsup_{|\Pi|\to 0} \max_{t\in\Pi} d^{p-q}(Z(t), Z(t_-)) \cdot V_q^q(Z) = 0.$$

■

Corollary 2.21. *If $p < 2$, then $V_p\left(B|_{[0,T]}\right) = \infty$ a.s.*

Proof. Choose partitions, $\{\Pi_m\}$, of $[0,T]$ such that $\lim_{m\to\infty} Q_m = T$ a.s. where $Q_m := V_2^2(B : \Pi_m)$ and let $\Omega_0 := \{\lim_{m\to\infty} Q_m = T\}$ so that $P(\Omega_0) = 1$. If $V_p\left(B|_{[0,T]}(\omega)\right) < \infty$ for then by Proposition 2.20,

$$\lim_{m\to\infty} Q_m(\omega) = \lim_{m\to\infty} V_2^2(B(\omega) : \Pi_m) = 0$$

and hence $\omega \notin \Omega_0$, i.e. $\left\{V_p\left(B|_{[0,T]}(\cdot)\right) < \infty\right\} \subset \Omega_0^c$. Therefore $\Omega_0 \subset \left\{V_p\left(B|_{[0,T]}(\cdot)\right) = \infty\right\}$ and hence

$$P\left(\left\{V_p\left(B|_{[0,T]}(\cdot)\right) = \infty\right\}\right) \geq P(\Omega) = 1.$$

■

Fact 2.22 *If $\{B_t\}_{t\geq 0}$ is a Brownian motion, then*

$$P(V_p(B) < \infty) = \begin{cases} 1 \text{ if } p > 2 \\ 0 \text{ if } p \leq 2 \end{cases}.$$

See for example [7, Exercise 1.14 on p. 36].

Corollary 2.23 (Roughness of Brownian Paths). *A Brownian motion, $\{B_t\}_{t\geq 0}$, is **not** almost surely α – Hölder continuous for any $\alpha > 1/2$.*

Proof. According to Proposition 2.19 we may choose partition, Π_m, such that mesh$(\Pi_m) \to 0$ and $Q_m \to T$ a.s. If B were α – Hölder continuous for some $\alpha > 1/2$, then

$$
\begin{aligned}
Q_m &= \sum_{t\in\Pi_m} (\Delta_t B)^2 \leq C \sum_{t\in\Pi_m} (\Delta t)^{2\alpha} \leq C \max\left([\Delta t]^{2\alpha-1}\right) \sum_{t\in\Pi_m} \Delta t \\
&\leq C\,[|\Pi_m|]^{2\alpha-1}\, T \to 0 \text{ as } m \to \infty
\end{aligned}
$$

which contradicts the fact that $Q_m \to T$ as $m \to \infty$. ■

2.3 The Bounded Variation Obstruction

Proposition 2.24. *Suppose that $Z(t)$ is a real continuous function such that $Z_0 = 0$ for simplicity. Define*

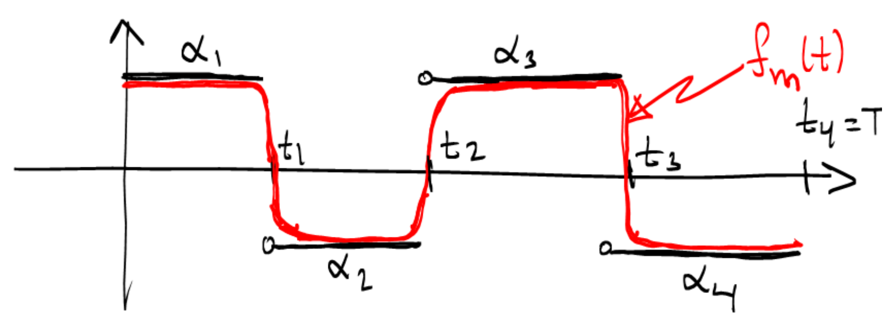
$$\int_0^T f(\tau)\, dZ(\tau) := -\int_0^T \dot{f}(\tau)\, Z(t)\, d\tau + f(t)\, Z(t)|_0^T$$

whenever f is a C^1 – function. If there exists, $C < \infty$ such that

$$\left|\int_0^T f(\tau)\, dZ(\tau)\right| \leq C \cdot \max_{0\leq\tau\leq T} |f(\tau)|, \tag{2.8}$$

then $V_1(Z) < \infty$ (See Definition 2.2 above) and the best possible choice for C in Eq. (2.8) is $V_1(Z)$.

Proof. Given a partition, $\Pi := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$, $\{\alpha_k\}_{k=1}^n \subset \mathbb{R}$, and $f(t) := \alpha_1 1_{\{0\}} + \sum_{k=1}^n \alpha_k 1_{(t_{k-1},t_k]}$. Choose $f_m(t)$ in $C^1([0,T], \mathbb{R})$ "well approximating" $f(t)$ as in Figure 2.3. It then is fairly



easy to show,

$$\int_0^T \dot{f}_m(\tau) Z(t) d\tau \to \sum_{k=1}^{n-1} (\alpha_{k+1} - \alpha_k) Z(t_k)$$

and therefore,

$$\lim_{m\to\infty} \int_0^T f_m(t) dZ(t) = -\sum_{k=1}^{n-1} (\alpha_{k+1} - \alpha_k) Z(t_k) + \alpha_n Z(t_n) - \alpha_1 Z(t_0)$$
$$= \sum_{k=1}^{n} \alpha_k (Z(t_k) - Z(t_{k-1})).$$

Therefore we have,

$$\left|\sum_{k=1}^{n} \alpha_k (Z(t_k) - Z(t_{k-1}))\right| = \lim_{m\to\infty} \left|\int_0^T f_m(\tau) dZ(\tau)\right|$$
$$\le C \cdot \limsup_{m\to\infty} \max_{0\le\tau\le T} |f_m(\tau)| = C \max_k |\alpha_k|.$$

Taking $\alpha_k = \operatorname{sgn}(Z(t_k) - Z(t_{k-1}))$ for each k, then shows $\sum_{k=1}^n |Z(t_k) - Z(t_{k-1})| \le C$. Since this holds for any partition Π, it follows that $V_1(Z) \le C$.

If $V_1(Z) < \infty$, then

$$\int_0^T \dot{f}(\tau) Z(t) d\tau = -\int_0^T f(t) d\lambda_Z(t) + f(t) Z(t)|_0^T$$

where λ_Z is the Lebesgue Stieltjes measure associated to Z. From this identity and integration by parts for such finite variation functions, it follows that

$$\int_0^T f(t) dZ(t) = \int_0^T f(t) d\lambda_Z(t)$$

and

$$\left|\int_0^T f(t) dZ(t)\right| = \left|\int_0^T f(t) d\lambda_Z(t)\right| \le \int_0^T |f(t)| d\|\lambda_Z\|(t)$$
$$\le \max_{0\le\tau\le T} |f(\tau)| \cdot \|\lambda_Z\|([0,T]) = V_1(Z) \cdot \max_{0\le\tau\le T} |f(\tau)|$$

Therefore C can be taken to be $V_1(Z)$ in Eq. (2.8) and hence $V_1(Z)$ is the best possible constant to use in this equation. ∎

Combining Fact 2.22 with Proposition 2.24 explains why we are going to have trouble defining $\int_0^t f_s dB_s$ when B is a Brownian motion. However, one might hope to use Young's integral in this setting.

Theorem 2.25 (L. C. Young 1936). *Suppose that $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} =: \theta > 1$. Then there exists a constant, $C(\theta) < \infty$ such that*

$$\left|\int_0^T f(t) dZ(t)\right| \le C(\theta) (\|f\|_\infty + V_q(f)) \cdot V_p(Z)$$

for all $f \in C^1$. Thus if $V_p(Z) < \infty$ the integral extends to those $f \in C([0,T])$ such that $V_q(f) < \infty$.

Unfortunately, Young's integral is still not sufficiently general to allow us to solve the typical SDE that we would like to consider. For example, consider the "simple" SDE,

$$\dot{y}(t) = B(t) \dot{B}(t) \text{ with } y(0) = 0.$$

The solution to this equation should be,

$$y(T) = \int_0^T B(t) dB(t)$$

which still does not make sense as a Young's integral when B is a Brownian motion because for any $p > 2$, $\frac{1}{p} + \frac{1}{p} =: \theta < 1$. For more on this point view see the very interesting work of Terry Lyons on "rough path analysis," [4].

2.4 Controls

Notation 2.26 (Controls) *Let*

$$\Delta = \{(s,t) : 0 \le s \le t \le T\}.$$

*A **control**, is a continuous function $\omega : \Delta \to [0,\infty)$ such that*

1. *$\omega(t,t) = 0$ for all $t \in [0,T]$,*
2. *ω is super-additive, i.e., for all $s \le t \le v$ we have*

$$\omega(s,t) + \omega(t,v) \le \omega(s,v). \tag{2.9}$$

Remark 2.27. If ω is a control then and $\omega(s,t)$ is increasing in t and decreasing in s for $(s,t) \in \Delta$. For example if $s \le \sigma \le t$, then $\omega(s,\sigma) + \omega(\sigma,t) \le \omega(s,t)$ and therefore, $\omega(\sigma,t) \le \omega(s,t)$. Similarly if $s \le t \le \tau$, then $\omega(s,t) + \omega(t,\tau) \le \omega(s,\tau)$ and therefore $\omega(s,t) \le \omega(s,\tau)$.

Lemma 2.28. *If ω is a control and $\varphi \in C([0,\infty) \to [0,\infty))$ such that $\varphi(0) = 0$ and φ is convex and increasing*[1]*, then $\varphi \circ \omega$ is also a control.*

[1] The assumption that φ is increasing is redundant here since we are assuming $\varphi'' \ge 0$ and we may deduce that $\varphi'(0) \ge 0$, it follows that $\varphi'(x) \ge 0$ for all x. This assertion also follows from Eq. (2.11).

Proof. We must show $\varphi \circ \omega$ is still superaditive. and this boils down to showing if $0 \le a, b, c$ with $a + b \le c$, then

$$\varphi(a) + \varphi(b) \le \varphi(c).$$

As φ is increasing, it suffices to show,

$$\varphi(a) + \varphi(b) \le \varphi(a+b). \tag{2.10}$$

Making use of the convexity of φ, we have,

$$\begin{aligned}\varphi(b) &= \varphi\left(\frac{a}{a+b}\cdot 0 + \frac{b}{a+b}(a+b)\right) \\ &\le \frac{a}{a+b}\varphi(0) + \frac{b}{a+b}\varphi(a+b) = \frac{b}{a+b}\varphi(a+b)\end{aligned}$$

and interchanging the roles of a and b gives,

$$\varphi(a) \le \frac{a}{a+b}\varphi(a+b). \tag{2.11}$$

.

Adding these last two inequalities then proves Eq. (2.10). ∎

Example 2.29. Suppose that $u(t)$ is any increasing continuous function of t, then $\omega(s,t) := u(t) - u(s)$ is a control which is in fact additive, i.e.

$$\omega(s,t) + \omega(t,v) = \omega(s,v) \text{ for all } s \le t \le v.$$

So for example $\omega(s,t) = t - s$ is an additive control and for any $p > 1$, $\omega(s,t) = (t-s)^p$ or more generally, $\omega(s,t) = (u(t) - u(s))^p$ is a control.

Lemma 2.30. *Suppose that ω is a control, $p \in [1,\infty)$, and $Z \in C([0,T], E)$ is a function satisfying,*

$$d(Z_s, Z_t) \le \omega(s,t)^{1/p} \text{ for all } (s,t) \in \Delta,$$

then $V_p^p(Z) \le \omega(0,T) < \infty$. More generally,

$$\omega_{p,Z}(s,t) := V_p^p\left(Z|_{[s,t]}\right) \le \omega(s,t) \text{ for all } (s,t) \in \Delta.$$

Proof. Let $(s,t) \in \Delta$ and $\Pi \in \mathcal{P}([s,t])$, then using the superadditivity of ω we find

$$V_p^p\left(Z|_{[s,t]} : \Pi\right) = \sum_{t\in\Pi} d^p\left(Z_t, Z_{t_-}\right) \le \sum_{t\in\Pi} \omega\left(Z_t, Z_{t_-}\right) \le \omega(s,t).$$

Therefore,

$$\omega_{p,Z}(s,t) := V_p^p\left(Z|_{[s,t]}\right) = \sup_{\Pi\in\mathcal{P}([s,t])} V_p^p\left(Z|_{[s,t]} : \Pi\right) \le \omega(s,t).$$

∎

Notation 2.31 *Given $o \in E$ and $p \in [1,\infty)$, let*

$$C_p([0,T], E) := \{Z \in C([0,T], E) : V_p(Z) < \infty\} \text{ and}$$

$$C_{0,p}([0,T], E) := \{Z \in C_p([0,T], E) : Z(0) = o\}.$$

Lemma 2.32. *Let $Z \in C_p([0,T], E)$ for some $p \in [1,\infty)$ and let $\omega := \omega_{Z,p} : \Delta \to [0,\infty)$ defined in Eq. (2.3). Then ω is superadditive. Furthermore if $p = 1$, ω is additive, i.e. Equality holds in Eq. (2.9).*

Proof. If $0 \le u \le s \le v \le T$ and $\Pi_1 \in \mathcal{P}(u,s)$, $\Pi_2 \in \mathcal{P}(s,v)$, then $\Pi_1 \cup \Pi_2 \in \mathcal{P}(u,v)$. Thus we have,

$$V_p^p(X : \Pi_1) + V_p^p(X : \Pi_2) = V_p^p(X : \Pi_1 \cup \Pi_2) \le \omega(u,v).$$

Taking the supremum over all Π_1 and Π_2 then implies,

$$\omega(u,s) + \omega(s,v) \le \omega(u,v) \text{ for all } u \le s \le v,$$

i.e. ω is superaditive.

In the case $p = 1$, it is easily seen using the triangle inequality that if $\Pi_1, \Pi_2 \in \mathcal{P}(s,t)$ and $\Pi_1 \subset \Pi_2$, then $V_1(X : \Pi_1) \le V_1(X, \Pi_2)$. Thus in comuting the sup of $V_1(X : \Pi)$ over all partition in $\mathcal{P}(s,t)$ it never hurts to add more points to a partition. Using this remark it is easy to show,

$$\begin{aligned}\omega(u,s) + \omega(s,v) &= \sup_{\Pi_1\in\mathcal{P}(u,s),\Pi_2\in\mathcal{P}(s,v)} [V_1(X : \Pi_1) + V_1(X : \Pi_2)] \\ &= \sup_{\Pi_1\in\mathcal{P}(u,s),\Pi_2\in\mathcal{P}(s,v)} V_1(X : \Pi_1 \cup \Pi_2) \\ &= \sup_{\Pi\in\mathcal{P}(u,v)} V_1(X : \Pi) = \omega(u,v)\end{aligned}$$

as desired. ∎

Lemma 2.33. *Let $Z \in C_p([0,T], E)$ for some $p \in [1,\infty)$, $\omega := \omega_{Z,p} : \Delta \to [0,\infty)$ defined in Eq. (2.3), $(a,b) \in \Delta$, $\Pi \in \mathcal{P}(a,b)$, and*

$$\varepsilon := \omega(a,b) - V_p^p(Z : \Pi) \ge 0.$$

Then for any $\Pi' \in \mathcal{P}(a,b)$ with $\Pi' \subset \Pi$, we have

$$\sum_{t\in\Pi'} \left[\omega(t_-,t) - V_p^p(Z : \Pi \cap [t_-,t])\right] \le \varepsilon. \tag{2.12}$$

In particular, if $(\alpha,\beta) \in \Delta \cap \Pi^2$ then

$$\omega(\alpha,\beta) \le V_p^p(Z : \Pi \cap [\alpha,\beta]) + \varepsilon. \tag{2.13}$$

Proof. Equation (2.12) is a simple consequence of the superadditivity of ω (Lemma 2.32) and the identity,

$$\sum_{t\in\Pi'} V_p^p\left(Z:\Pi\cap[t_-,t]\right)=V_p^p(Z:\Pi).$$

Indeed, using these properties we find,

$$\begin{aligned}\sum_{t\in\Pi'}\left[\omega\left(t_-,t\right)-V_p^p\left(Z:\Pi\cap[t_-,t]\right)\right]&=\sum_{t\in\Pi'}\omega\left(t_-,t\right)-V_p^p(Z:\Pi)\\&\le\omega\left(a,b\right)-V_p^p(Z:\Pi)=\varepsilon.\end{aligned}$$

■

Lemma 2.34. *Suppose that $Z\in C_p([0,T],E)$ for some $p\in[1,\infty)$ and $\varepsilon>0$ is given. Then there exists $\delta>0$ such that, for every $\Pi\subset\subset[0,T]$ and $u\in[0,T]$ such that* $\operatorname{dist}(u,\Pi)<\delta$ *we have,*

$$\left|V_p^p\left(Z:\Pi\right)-V_p^p\left(Z:\Pi\cup\{u\}\right)\right|<\varepsilon.$$

Proof. Let $\rho(s,t):=d^p\left(Z(s),Z(t)\right)$ and choose (by the uniform continuity of ρ) $\delta>0$ such that $|\rho(s,t)-\rho(u,v)|<\varepsilon/2$ provided $|(s,t)-(u,v)|<\delta$. Suppose that $\Pi=\{t_0<t_1<\dots<t_n\}\subset[0,T]$ and $u\in[0,T]$ such that $\operatorname{dist}(u,\Pi)<\delta$. There are now three case to consider, $u\in(t_0,t_n)$, $u<t_0$ and $u>t_1$. In the first case, suppose that $t_{i-1}<u<t_i$ and that (for the sake of definiteness) that $|t_i-u|<\delta$, then

$$\begin{aligned}\left|V_p^p\left(Z:\Pi\right)-V_p^p\left(Z:\Pi\cup\{u\}\right)\right|&=\left|\rho\left(t_{i-1},t_i\right)-\rho\left(t_{i-1},u\right)-\rho\left(u,t_i\right)\right|\\&\le\left|\rho\left(t_{i-1},t_i\right)-\rho\left(t_{i-1},u\right)\right|+\left|\rho\left(u,t_i\right)-\rho\left(t_i,t_i\right)\right|<\varepsilon.\end{aligned}$$

The second and third case are similar. For example if $u<t_0$, we will have,

$$\left|V_p^p\left(Z:\Pi\cup\{u\}\right)-V_p^p\left(Z:\Pi\right)\right|=\rho\left(u,t_0\right)=\rho\left(u,t_0\right)-\rho\left(t_0,t_0\right)<\varepsilon/2.$$

■

Theorem 2.35 (The variation control). *Let $p\in[1,\infty)$ and suppose that $Z\in C_p([0,T],E)$. Then $\omega_{Z,p}:\Delta\to[0,\infty)$ defined in Eq. (2.3) is a control satisfying, $d\left(Z(s),Z(t)\right)\le\omega_{Z,p}(s,t)^{1/p}$ for all $(s,t)\in\Delta$.*

Proof. Let $\omega(s,t):=\omega_{Z,p}(s,t)$ and $\rho(s,t):=d^p\left(Z(s),Z(t)\right)$. It is clear by the definition of ω, the $\omega(t,t)=0$ for all t and we have already seen in Lemma 2.32 that ω is superadditive. So to finish the proof we must show ω is continuous.

Using Remark 2.27, we know that $\omega(s,t)$ is increasing in t and decreasing in s and therefore $\omega(u+,v-)=\lim_{s\downarrow u,t\uparrow v}\omega(s,t)$ and $\omega(u-,v+)=\lim_{s\uparrow u,,t\downarrow v}\omega(s,t)$ exists and satisfies,

$$\omega\left(u+,v-\right)\le\omega\left(u,v\right)\le\omega\left(u-,v+\right). \tag{2.14}$$

The main crux of the continuity proof is to show that the inequalities in Eq. (2.14) are all equalities.

1. Suppose that $\varepsilon>0$ is given and $\delta>0$ is chosen as in Lemma 2.34 and suppose that $u<s<t<v$ with $|s-u|<\delta$ and $|v-t|<\delta$. Further let $\Pi\in\mathcal{P}(u,v)$ be a partition of $[u,v]$, then according to Lemma 2.34,

$$\begin{aligned}V_p^p\left(Z:\Pi\right)&\le V_p^p\left(Z:\Pi\cup\{s,t\}\right)+2\varepsilon\\&=\rho\left(u,s\right)+\rho\left(t,v\right)+V_p^p\left(Z:\Pi\cap[s,t]\cup\{s,t\}\right)+2\varepsilon\\&\le\rho\left(u,s\right)+\rho\left(t,v\right)+\omega\left(s,t\right)+2\varepsilon.\end{aligned}$$

Letting $s\downarrow u$ and $t\uparrow v$ in this inequality shows,

$$V_p^p\left(Z:\Pi\right)\le\omega\left(u+,v-\right)+2\varepsilon$$

and then taking the supremum over $\Pi\in\mathcal{P}(u,v)$ and then letting $\varepsilon\downarrow 0$ shows $\omega(u,v)\le\omega(u+,v-)$. Combined this with the first inequality in Eq. (2.14) shows, $\omega(u+,v-)=\omega(u,v)$.

2. We will now show $\omega(u,v)=\omega(u-,v+)$ by showing $\omega(u-,v+)\le\omega(u,v)$. Let $\varepsilon>0$ and $\delta>0$ be as in Lemma 2.34 and suppose that $s<u$ and $t>v$ with $|u-s|<\delta$ and $|t-v|<\delta$. Let us now choose a partition $\Pi\in\mathcal{P}(s,t)$ such that

$$\omega\left(s,t\right)\le V_p^p\left(Z:\Pi\right)+\varepsilon.$$

Then applying Lemma 2.34 gives,

$$\omega\left(s,t\right)\le V_p^p\left(Z:\Pi_1\right)+3\varepsilon$$

where $\Pi_1=\Pi\cup\{u,v\}$. As above, let u_- and v_+ be the elements in Π_1 just before u and just after v respectively. An application of Lemma 2.33 then shows,

$$\begin{aligned}\omega\left(u-,v+\right)&\le\omega\left(u_-,v_+\right)\le V_p^p\left(Z:\Pi_1\cap[u_-,v_+]\right)+3\varepsilon\\&=V_p^p\left(Z:\Pi_1\right)+\rho\left(u_-,u\right)+\rho\left(v,v_+\right)+3\varepsilon \qquad \boxed{\Pi_1\cap[u,v]}\\&\le\omega\left(u,v\right)+5\varepsilon.\end{aligned}$$

As $\varepsilon>0$ was arbitrary we may conclude $\omega(u-,v+)\le\omega(u,v)$ which completes the proof that $\omega(u-,v+)=\omega(u,v)$.

I now claim all the other limiting directions follow easily from what we have proved. For example,

$$\omega(u,v) \le \omega(u,v+) \le \omega(u-,v+) = \omega(u,v) \implies \omega(u,v+) = \omega(u,v),$$
$$\omega(u,v) = \omega(u+,v-) \le \omega(u,v-) \le \omega(u,v) \implies \omega(u,v-) = \omega(u,v),$$

and similarly, $\omega(u\pm,v) = \omega(u,v)$. We also have,

$$\omega(u,v) = \omega(u+,v-) \le \liminf_{s\downarrow u,\ t\downarrow v} \omega(s,t) \le \limsup_{s\downarrow u,\ t\downarrow v} \omega(s,t) \le \omega(u-,v+) = \omega(u,v)$$

which shows $\omega(u+,v+) = \omega(u,v)$ and

$$\omega(u,v) = \omega(u+,v-) \le \liminf_{s\uparrow u,\ t\uparrow v} \omega(s,t) \le \liminf_{s\uparrow u,\ t\uparrow v} \omega(s,t) \le \omega(u-,v+) = \omega(u,v)$$

so that $\omega(u-,v-) = \omega(u,v)$. ■

Proposition 2.36 (See [2, Proposition 5.15 from p. 83.]). *Let (E,d) be a metric space, and let $x:[0,T]\to E$ be a continuous path. Then x is of finite p-variation if and only if there exists a continuous increasing (i.e. non – decreasing) function $h:[0,T]\to\left[0,V_p^p(Z)\right]$ and a $1/p$ – Hölder path $g:\left[0,V_p^p(Z)\right]\to E$ such that such that $x=g\circ h$. More explicitly we have,*

$$d(g(v),g(u)) \le |v-u|^{1/p} \text{ for all } u,v\in\left[0,V_p^p(Z)\right]. \tag{2.15}$$

Proof. Let $\omega(s,t) := \omega_{p,x}(s,t) = V_p^p\left(x|_{[s,t]}\right)$ be the control associated to x and define $h(t) := \omega(0,t)$. Observe that h is increasing and for $0\le s\le t\le T$ that $h(s)+\omega(s,t)\le h(t)$, i.e.

$$\omega(s,t) \le h(t)-h(s) \text{ for all } 0\le s\le t\le T.$$

Let $g:[0,h(T)]\to E$ be defined by $g(h(t)) := x(t)$. This is well defined since if $s\le t$ and $h(s)=h(t)$, then $\omega(s,t)=0$ and hence $x|_{[s,t]}$ is constant and in particular $x(s)=x(t)$. Moreover it now follows for $s<t$ such that $u:=h(s)<h(t)=:v$, that

$$\begin{aligned} d^p(g(v),g(u)) &= d^p(g(h(t)),g(h(s))) = d^p(x(t),x(s)) \\ &\le \omega(s,t) \le h(t)-h(s) = v-u \end{aligned}$$

from which Eq. (2.15) easily follows. ■

3

The Bounded Variation Theory

3.1 Integration Theory for Simple Functions

Let $T \in (0, \infty)$ be fixed,

$$\mathcal{S} := \{(a, b] : 0 \le a \le b \le T\} \cup \{[0, b] \cap \mathbb{R} : 0 \le b \le T\}. \tag{3.1}$$

Further let $\mathcal{A}$ be the algebra generated by $\mathcal{S}$. Since $\mathcal{S}$ is an elementary set, $\mathcal{A}$ may be described as the collection of sets which are finite disjoint unions of subsets from $\mathcal{S}$. Given any function, $Z : [0, T] \to V$ with V being a vector define $\mu_Z : \mathcal{S} \to V$ via,

$$\mu_Z((a, b]) := Z_b - Z_a \text{ and } \mu_Z([0, b]) = Z_b - Z_0 \;\forall\; 0 \le a \le b \le T.$$

Lemma 3.1. *μ_Z is finitely additive on $\mathcal{S}$ and hence extends to a finitely additive measure on $\mathcal{A}$.*

Proof. See Chapter **??** and in particular make the minor necessary modifications to Examples **??**, **??**, and Proposition **??**. ■

Let W be another vector space and $f : [0, T] \to \operatorname{End}(V, W)$ be an $\mathcal{A}$ – **simple function**, i.e. $f([0, T])$ is a finite set and $f^{-1}(\lambda) \in \mathcal{A}$ for all $\lambda \in \operatorname{End}(V, W)$. For such functions we define,

$$\int_{[0,T]} f(t)\, dZ(t) := \int_{[0,T]} f d\mu_Z = \sum_{\lambda \in \operatorname{End}(V,W)^\times} \lambda \mu_Z(f = \lambda) \in W. \tag{3.2}$$

The basic linearity properties of this integral are explained in Proposition **??**. For later purposes, it will be useful to have the following substitution formula at our disposal.

Theorem 3.2 (Substitution formula). *Suppose that f and Z are as above and $Y_t = \int_{[0,t]} f d\mu_Z \in W$. Further suppose that $g : \mathbb{R}_+ \to \operatorname{End}(W, U)$ is another $\mathcal{A}$ – simple function with finite support. Then*

$$\int_{[0,T]} g d\mu_Y = \int_{[0,T]} g f d\mu_Z.$$

Proof. By definition of these finitely additive integrals,

$$\begin{aligned}\mu_Y((a, b]) = Y_b - Y_a &= \int_{[0,b]} f d\mu_Z - \int_{[0,a]} f d\mu_Z \\ &= \int_{[0,T]} \left(1_{[0,b]} - 1_{[0,a]}\right) f d\mu_Z = \int_{[0,T]} 1_{(a,b]} f d\mu_Z.\end{aligned}$$

Therefore, it follows by the finite additivity of μ_Y and linearity $\int_{[0,T]} (\cdot)\, d\mu_Z$, that

$$\mu_Y(A) = \int_A f d\mu_Z = \int_{[0,T]} 1_A f d\mu_Z \text{ for all } A \in \mathcal{A}.$$

Therefore,

$$\begin{aligned}\int_{[0,T]} g d\mu_Y &= \sum_{\lambda \in \operatorname{End}(W,U)^\times} \lambda \mu_Y(g = \lambda) = \sum_{\lambda \in \operatorname{End}(W,U)^\times} \lambda \int_{[0,T]} 1_{\{g=\lambda\}} f d\mu_Z \\ &= \int_{[0,T]} \sum_{\lambda \in \operatorname{End}(W,U)^\times} 1_{\{g=\lambda\}} \lambda f d\mu_Z = \int_{[0,T]} g f d\mu_Z\end{aligned}$$

as desired. ■

Let us observe that

$$\left\| \int_{[0,T]} f(t)\, dZ_t \right\| \le \sum_{\lambda \in \operatorname{End}(V,W)} \|\lambda\| \left\|\mu_Z(f = \lambda)\right\|.$$

Let us now define,

$$\begin{aligned}\|\mu_Z\|((a, b]) &:= V_1\left(Z|_{[a,b]}\right) \\ &= \sup\left\{\sum_{j=1}^n \left\|Z_{t_j} - Z_{t_{j-1}}\right\| : a = t_0 < t_1 < \cdots < t_n = b \text{ and } n \in \mathbb{N}\right\}\end{aligned}$$

be the variation measure associated to μ_Z.

Lemma 3.3. *If $\|\mu_Z\|((0, T]) < \infty$, then $\|\mu_Z\|$ is a finitely additive measure on $\mathcal{S}$ and hence extends to a finitely additive measure on $\mathcal{A}$.*

Proof. The additivity on $\mathcal{S}$ was already verified in Lemma 2.32. Here is the proof again for sake of convenience.

Suppose that $\Pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$, $s \in (t_{l-1}, t_l)$ for some l, and $\Pi' := \Pi \cup \{s\}$. Then

$$\begin{aligned}\|\mu_Z\|^{\Pi}((a,b]) &:= \sum_{j=1}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| \\ &= \sum_{j=1:j\neq l}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| + \left\|Z_{t_l} - Z_s + Z_s - Z_{t_{l-1}}\right\| \\ &\le \sum_{j=1:j\neq l}^{n} \left\|Z_{t_j} - Z_{t_{j-1}}\right\| + \left\|Z_{t_l} - Z_s\right\| + \left\|Z_s - Z_{t_{l-1}}\right\| \\ &= \|\mu_Z\|^{\Pi'}((a,b]) \le \|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]).\end{aligned}$$

Hence it follows that

$$\|\mu_Z\|((a,b]) = \sup_{\Pi} \|\mu_Z\|^{\Pi}((a,b]) \le \|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]).$$

Conversely if Π_1 is a partition of $(a,s]$ and Π_2 is a partition of $(s,b]$, then $\Pi := \Pi_1 \cup \Pi_2$ is a partition of $(a,b]$. Therefore,

$$\|\mu_Z\|^{\Pi_1}((a,s]) + \|\mu_Z\|^{\Pi_2}((s,b]) = \|\mu_Z\|^{\Pi}((a,b]) \le \|\mu_Z\|((a,b])$$

and therefore,

$$\|\mu_Z\|((a,s]) + \|\mu_Z\|((s,b]) \le \|\mu_Z\|((a,b]).$$

■

Corollary 3.4. *If Z has finite variation on $[0,T]$, then we have*

$$\left\|\int_{[0,T]} f(t)\, dZ_t\right\| \le \int_{[0,T]} \|f(\lambda)\| \, \|\mu_Z\|(d\lambda) \le \|f\|_\infty \cdot \|\mu_Z\|([0,T]).$$

Proof. Simply observe that $\|\mu_Z(A)\| \le \|\mu_Z\|(A)$ for all $A \in \mathcal{A}_T$ and hence from Eq. (3.2) we have

$$\begin{aligned}\left\|\int_{[0,T]} f(t)\, dZ_t\right\| &\le \sum_{\lambda\in\mathrm{End}(V,W)} \|\lambda\| \, \|\mu_Z(f=\lambda)\| \\ &\le \sum_{\lambda\in\mathrm{End}(V,W)} \|\lambda\| \, \|\mu_Z\|(f=\lambda) = \int_{[0,T]} \|f(\lambda)\| \, \|\mu_Z\|(d\lambda) \\ &\le \|f\|_\infty \cdot \|\mu_Z\|([0,T]).\end{aligned}$$

■

Notation 3.5 *In the future we will often write $\|dZ\|$ for $d\|\mu_Z\|$.*

Thus is we are in the Banach space setting and Z has finite variation on $[0,T]$ we may define the integral, $\int_{[0,T]} f(t)\, dZ_t$ for any function f which is in the uniform closure of the $\mathrm{End}(V,W)$ – valued simple functions. This space contains all of the continuous functions, $f : [0,T] \to \mathrm{End}(V,W)$.

Exercise 3.1 (Fundamental Theorem of Calculus). Prove the fundamental theorem of calculus in this context. That is; if $f : V \to W$ be a C^1 – function and $\{Z_t\}_{t\ge 0}$ is a V – valued function of locally bounded variation, then for all $0 \le a < b \le T$,

$$f(Z_b) - f(Z_a) = \int_a^b f'(Z_\tau)\, dZ_\tau := \int_{[a,b]} f'(Z_\tau)\, dZ_\tau,$$

where $f'(z) \in \mathrm{End}(V,W)$ is defined by, $f'(z)\, v := \frac{d}{dt}|_0 f(z+tv)$.

Solution to Exercise (3.1). Let $\Pi \in \mathcal{P}(0,T)$. Then by a telescoping series argument,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} \Delta_t f(Z.)$$

where

$$\begin{aligned}\Delta_t f(Z.) &= f(Z_t) - f\left(Z_{t_-}\right) = f\left(Z_{t_-} + \Delta_t Z\right) - f\left(Z_{t_-}\right) \\ &= \int_0^1 f'\left(Z_{t_-} + s\Delta_t Z\right) \Delta_t Z \, ds = f'\left(Z_{t_-}\right)\Delta_t Z + \varepsilon_t^{\Pi} \Delta_t Z\end{aligned}$$

and

$$\varepsilon_t^{\Pi} := \int_0^1 \left[f'\left(Z_{t_-} + s\Delta_t Z\right) - f'\left(Z_{t_-}\right)\right] \, ds.$$

Thus we have,

$$f(Z_b) - f(Z_a) = \sum_{t\in\Pi} f'\left(Z_{t_-}\right)\Delta_t Z + \delta_\Pi = \int_{[a,b]} f'\left(Z_{t_-}\right) dZ(t) + \delta_\Pi \tag{3.3}$$

where $\delta_\Pi := \sum_{t\in\Pi} \varepsilon_t^{\Pi} \Delta_t Z$. Since,

$$\begin{aligned}\|\delta_\Pi\| &\le \sum_{t\in\Pi} \left\|\varepsilon_t^{\Pi} \Delta_t Z\right\| \le \sum_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \|\Delta_t Z\| \le \max_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \cdot \sum_{t\in\Pi} \|\Delta_t Z\| \\ &\le \max_{t\in\Pi} \left\|\varepsilon_t^{\Pi}\right\| \cdot V_1(Z),\end{aligned}$$

and

$$\left\|\varepsilon_t^{\Pi}\right\| := \int_0^1 \left\|\left[f'\left(Z_{t_-} + s\Delta_t Z\right) - f'\left(Z_{t_-}\right)\right]\right\| ds.$$

Since $g(s,\tau,t) := \|[f'(Z_\tau + s(Z_t - Z_\tau)) - f'(Z_\tau)]\|$ is a continuous function in $s \in [0,1]$ and $\tau, t \in [0,T]$ with $g(s,t,t) = 0$ for all s and t, it follows by uniform continuity arguments that $g(s,\tau,t)$ is small whenever $|t-\tau|$ is small. Therefore, $\lim_{|\Pi|\to 0}\left\|\varepsilon_t^{\Pi}\right\| = 0$. Moreover, again by a uniform continuity argument, $f'\left(Z_{t_-}\right) \to f'(Z_t)$ uniformly as $|\Pi| \to 0$. Thus we may pass to the limit as $|\Pi| \to 0$ in Eq. (3.3) to complete the proof.

3.2 Calculus Bounds

For the exercises to follow we suppose that μ is a positive σ − finite measure on $\left([0,\infty), \mathcal{B}_{[0,\infty)}\right)$ such that μ is continuous, i.e. $\mu(\{s\}) = 0$ for all $s \in [0,\infty)$. We will further write,

$$\int_0^t f(s)\,d\mu(s) := \int_{[0,t]} f(s)\,d\mu(s) = \int_{(0,t)} f(s)\,d\mu(s),$$

wherein the second equality holds since μ is continuous. Although it is not necessary, you may use Exercise 3.1 with $Z_t := \mu([0,t])$ to solve the following problems.

Exercise 3.2. Show for all $0 \le a < b < \infty$ and $n \in \mathbb{N}$ that

$$h_n(b) := \int_{a \le s_1 \le s_2 \le \dots \le s_n \le b} d\mu(s_1)\dots d\mu(s_n) = \frac{\mu([a,b])^n}{n!}. \tag{3.4}$$

Solution to Exercise (3.2). First solution. Let us observe that $h(t) := h_1(t) = \mu([a,t])$ and $h_n(t)$ satisfies the recursive relation,

$$h_{n+1}(t) := \int_a^t h_n(s)\,d\mu(s) = \int_a^t h_n(s)\,dh(s) \text{ for all } t \ge a.$$

Now let $H_n(t) := \frac{1}{n!}h^n(t)$, by an application of Exercise 3.1 with $f(x) = x^{n+1}/(n+1)!$ implies,

$$H_{n+1}(t) = H_{n+1}(t) - H_{n+1}(a) = \int_a^t f'(h(\tau))\,dh(\tau) = \int_a^t H_n(\tau)\,dh(\tau)$$

and therefore it follows that $H_n(t) = h_n(t)$ for all $t \ge a$ and $n \in \mathbb{N}$.

Second solution. If $i \ne j$, it follows by Fubini's theorem that

$$\begin{aligned}
&\mu^{\otimes n}\left(\{(s_1,\dots,s_n) \in [a,b]^n : s_i = s_j\}\right) \\
&\quad = \mu([a,b])^{n-2} \cdot \int_{[a,b]^2} 1_{s_i = s_j}\,d\mu(s_i)\,d\mu(s_j) \\
&\quad = \mu([a,b])^{n-2} \cdot \int_{[a,b]} \mu(\{s_j\})\,d\mu(s_j) = 0.
\end{aligned}$$

From this observation it follows that

$$1_{[a,b]^n}(s_1,\dots,s_n) = \sum_{\sigma \in \mathcal{S}_n} 1_{a \le s_{\sigma 1} \le s_{\sigma 2} \le \dots \le s_{\sigma n} \le b} - \mu^{\otimes n} - \text{a.e.},$$

where σ ranges over the permutations, $\mathcal{S}_n$, of $\{1,2,\dots,n\}$. Integrating this equation relative with respect to $\mu^{\otimes n}$ and then using Fubini's theorem gives,

$$\begin{aligned}
\mu([a,b])^n = \mu^{\otimes n}([a,b]^n) &= \sum_{\sigma \in \mathcal{S}_n} \int 1_{a \le s_{\sigma 1} \le s_{\sigma 2} \le \dots \le s_{\sigma n} \le b}\,d\mu^{\otimes n}(\mathbf{s}) \\
&= \sum_{\sigma \in \mathcal{S}_n} \int 1_{a \le s_{\sigma 1} \le s_{\sigma 2} \le \dots \le s_{\sigma n} \le b}\,d\mu(s_1)\dots d\mu(s_n) \\
&= \sum_{\sigma \in \mathcal{S}_n} \int_{a \le s_1 \le s_2 \le \dots \le s_n \le b} d\mu(s_1)\dots d\mu(s_n) \\
&= n! \int_{a \le s_1 \le s_2 \le \dots \le s_n \le b} d\mu(s_1)\dots d\mu(s_n).
\end{aligned}$$

Exercise 3.3 (Gronwall's Lemma). If $\varepsilon(t)$ and $f(t)$ are continuous non-negative functions such that

$$f(t) \le \varepsilon(t) + \int_0^t f(\tau)\,d\mu(\tau), \tag{3.5}$$

then

$$f(t) \le \varepsilon(t) + \int_0^t e^{\mu([\tau,t])}\varepsilon(\tau)\,d\mu(\tau). \tag{3.6}$$

If we further assume that ε is increasing, then

$$f(t) \le \varepsilon(t)\,e^{\mu([0,t])}. \tag{3.7}$$

Solution to Exercise (3.3). Feeding Eq. (3.5) back into itself implies

$$
\begin{aligned}
f(t) &\le \varepsilon(t) + \int_0^t \left[\varepsilon(\tau) + \int_0^\tau f(s)\, d\mu(s)\right] d\mu(\tau) \\
&= \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0 \le s_2 \le s_1 \le t} f(s_2)\, d\mu(s_1)\, d\mu(s_2) \\
&\le \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0 \le s_2 \le s_1 \le t} \left[\varepsilon(s_2) + \int_0^{s_2} f(s_3)\, d\mu(s_3)\right] d\mu(s_1)\, d\mu(s_2) \\
&= \varepsilon(t) + \int_0^t \varepsilon(s_1)\, d\mu(s_1) + \int_{0 \le s_2 \le s_1 \le t} \varepsilon(s_2)\, d\mu(s_1)\, d\mu(s_2) \\
&\quad + \int_{0 \le s_3 \le s_2 \le s_1 \le t} f(s_3)\, d\mu(s_1)\, d\mu(s_2)\, d\mu(s_3).
\end{aligned}
$$

Continuing in this manner inductively shows,

$$
f(t) \le \varepsilon(t) + \sum_{k=1}^N \int_{0 \le s_k \le \dots \le s_2 \le s_1 \le t} \varepsilon(s_k)\, d\mu(s_1) \dots d\mu(s_k) + R_N(t) \tag{3.8}
$$

where, using Exercise 3.2,

$$
\begin{aligned}
R_N(t) &:= \int_{0 \le s_{k+1} \le \dots \le s_2 \le s_1 \le t} f(s_{k+1})\, d\mu(s_1) \dots d\mu(s_k)\, d\mu(s_{k+1}) \\
&\le \max_{0 \le s \le t} f(t) \cdot \frac{\mu([0,t])^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty.
\end{aligned}
$$

So passing to the limit in Eq. (3.8) and again making use of Exercise 3.2 shows,

$$
\begin{aligned}
f(t) &\le \varepsilon(t) + \sum_{k=1}^\infty \int_{0 \le s_k \le \dots \le s_2 \le s_1 \le t} \varepsilon(s_k)\, d\mu(s_1) \dots d\mu(s_k) \qquad (3.9) \\
&= \varepsilon(t) + \sum_{k=1}^\infty \int_0^t \varepsilon(s_k) \frac{\mu([s_k,t])^{k-1}}{(k-1)!} d\mu(s_k) \\
&= \varepsilon(t) + \int_0^t \varepsilon(\tau) \cdot \sum_{k=1}^\infty \frac{\mu([\tau,t])^{k-1}}{(k-1)!} d\mu(\tau) \\
&= \varepsilon(t) + \int_0^t e^{\mu([\tau,t])} \varepsilon(\tau)\, d\mu(\tau).
\end{aligned}
$$

If we further assume that ε is increasing, then from Eq. (3.9) and Exercise 3.2 we have

$$
\begin{aligned}
f(t) &\le \varepsilon(t) + \varepsilon(t) \sum_{k=1}^\infty \int_{0 \le s_k \le \dots \le s_2 \le s_1 \le t} d\mu(s_1) \dots d\mu(s_k) \\
&= \varepsilon(t) + \varepsilon(t) \sum_{k=1}^\infty \frac{\mu([0,t])^k}{k!} = \varepsilon(t)\, e^{\mu([0,t])}.
\end{aligned}
$$

Alternatively if we let $Z_t := \mu([0,t])$, then

$$
\begin{aligned}
\int_0^t e^{\mu([\tau,t])} d\mu(\tau) &= \int_0^t e^{Z_t - Z_\tau} dZ_\tau = \int_0^t d_\tau \left(-e^{Z_t - Z_\tau}\right) \\
&= \left(-e^{Z_t - Z_\tau}\right)_0^t = e^{Z_t} - 1.
\end{aligned}
$$

Therefore,

$$
f(t) \le \varepsilon(t) + \varepsilon(t)\left(e^{Z_t} - 1\right) = \varepsilon(t)\, e^{Z_t}.
$$

Exercise 3.4. Suppose that $\{\varepsilon_n(t)\}_{n=0}^\infty$ is a sequence of non-negative continuous functions such that

$$
\varepsilon_{n+1}(t) \le \int_0^t \varepsilon_n(\tau)\, d\mu(\tau) \text{ for all } n \ge 0 \tag{3.10}
$$

and $\delta(t) = \max_{0 \le \tau \le t} \varepsilon_0(\tau)$. Show

$$
\varepsilon_n(t) \le \delta(t) \frac{\mu([0,t])^n}{n!} \text{ for all } n \ge 0.
$$

Solution to Exercise (3.4). By iteration of Eq. (3.10) we find,

$$
\begin{aligned}
\varepsilon_1(t) &\le \int_0^t \varepsilon_0(\tau)\, d\mu(\tau) \le \delta(t) \int_{0 \le s_1 \le t} d\mu(s_1), \\
\varepsilon_2(t) &\le \int_0^t \varepsilon_1(s_2)\, d\mu(s_2) \le \delta(t) \int_0^t \left[\int_{0 \le s_1 \le t} d\mu(s_1)\right] d\mu(s_2) \\
&= \delta(t) \int_{0 \le s_2 \le s_1 \le t} d\mu(s_1)\, d\mu(s_2), \\
&\vdots \\
\varepsilon_n(t) &\le \delta(t) \int_{0 \le s_n \le \dots \le s_1 \le t} d\mu(s_1) \dots d\mu(s_n).
\end{aligned}
$$

The result now follows directly from Exercise 3.2.

3.3 Bounded Variation Ordinary Differential Equations

In this section we begin by reviewing some of the basic theory of ordinary differential equations – O.D.E.s for short. Throughout this chapter we will let X and Y be Banach spaces, $U \subset_o Y$ an open subset of Y, and $y_0 \in U$, $x : [0,T] \to X$ is a continuous process of bounded variation, and $F : [0,T] \times U \to \mathrm{End}(X,Y)$ is a continuous function. (We will make further assumptions on F as we need them.) Our goal here is to investigate the "ordinary differential equation,"

$$\dot{y}(t) = F(t, y(t))\,\dot{x}(t) \text{ with } y(0) = y_0 \in U. \tag{3.11}$$

Since x is only of bounded variation, to make sense of this equation we will interpret it in its integrated form,

$$y(t) = y_0 + \int_0^t F(\tau, y(\tau))\,dx(\tau). \tag{3.12}$$

Proposition 3.6 (Continuous dependence on the data). *Suppose that $G : [0,T] \times U \to \mathrm{End}(X,Y)$ is another continuous function, $z : [0,T] \to X$ is another continuous function with bounded variation, and $w : [0,T] \to U$ satisfies the differential equation,*

$$w(t) = w_0 + \int_0^t G(\tau, w(\tau))\,dz(\tau) \tag{3.13}$$

for some $w_0 \in U$. Further assume there exists a continuous function, $K(t) \geq 0$ such that F satisfies the ***Lipschitz condition,***

$$\|F(t,y) - F(t,w)\| \leq K(t)\,\|y - w\| \text{ for all } 0 \leq t \leq T \text{ and } y, w \in U. \tag{3.14}$$

Then

$$\|y(t) - w(t)\| \leq \varepsilon(t) \exp\left(\int_0^t K(\tau)\,\|dx(\tau)\|\right). \tag{3.15}$$

where

$$\varepsilon(t) := \|y_0 - w_0\| + \int_0^t \|F(\tau, w(\tau)) - G(\tau, w(\tau))\|\,\|dx(\tau)\| + \int_0^t \|G(\tau, w(\tau))\|\,\|d(x - z)(\tau)\| \tag{3.16}$$

Proof. Let $\delta(t) := y(t) - w(t)$, so that $y = w + \delta$. We then have,

$$\begin{aligned}
\delta(t) &= y_0 - w_0 + \int_0^t F(\tau, y(\tau))\,dx(\tau) - \int_0^t G(\tau, w(\tau))\,dz(\tau) \\
&= y_0 - w_0 + \int_0^t F(\tau, w(\tau) + \delta(\tau))\,dx(\tau) - \int_0^t G(\tau, w(\tau))\,dz(\tau) \\
&= y_0 - w_0 + \int_0^t [F(\tau, w(\tau)) - G(\tau, w(\tau))]\,dx(\tau) + \int_0^t G(\tau, w(\tau))\,d(x - z)(\tau) \\
&+ \int_0^t [F(\tau, w(\tau) + \delta(\tau)) - F(\tau, w(\tau))]\,dx(\tau).
\end{aligned}$$

Crashing through this identity with norms shows,

$$\|\delta(t)\| \leq \varepsilon(t) + \int_0^t K(\tau)\,\|\delta(\tau)\|\,\|dx(\tau)\|$$

where $\varepsilon(t)$ is given in Eq. (3.16). The estimate in Eq. (3.15) is now a consequence of this inequality and Exercise 3.3 with $d\mu(\tau) := K(\tau)\,\|dx(\tau)\|$. ■

Corollary 3.7 (Uniquness of solutions). *If F satisfies the Lipschitz hypothesis in Eq. (3.14), then there is at most one solution to the ODE in Eq. (3.12).*

Proof. Simply apply Proposition 3.6 with $F = G$, $y_0 = w_0$, and $x = z$. In this case $\varepsilon \equiv 0$ and the result follows. ■

Proposition 3.8 (An apriori growth bound). *Suppose that $U = Y$, $T = \infty$, and there are continuous functions, $a(t) \geq 0$ and $b(t) \geq 0$ such that*

$$\|F(t,y)\| \leq a(t) + b(t)\,\|y\| \text{ for all } t \geq 0 \text{ and } y \in Y.$$

Then

$$\|y(t)\| \leq \left(\|y_0\| + \int_0^t a(\tau)\,d\nu(\tau)\right) \exp\left(\int_0^t b(\tau)\,d\nu(\tau)\right). \tag{3.17}$$

Proof. Let $\nu(t) := \omega_{x,1}(0,t) = \|s\|_{1\text{-Var}}(t)$. From Eq. (3.12) we have,

$$\begin{aligned}
\|y(t)\| &\leq \|y_0\| + \int_0^t \|F(\tau, y(\tau))\|\,d\nu(\tau) \\
&\leq \|y_0\| + \int_0^t (a(\tau) + b(\tau)\,\|y(\tau)\|)\,d\nu(\tau) \\
&= \varepsilon(t) + \int_0^t \|y(\tau)\|\,d\mu(\tau)
\end{aligned}$$

where

$$\varepsilon(t) := \|y_0\| + \int_0^t a(\tau)\,d\nu(\tau) \text{ and } d\mu(\tau) := b(\tau)\,d\nu(\tau).$$

Hence we may apply Exercise 3.3 to learn $\|y(t)\| \leq \varepsilon(t)\,e^{\mu([0,t])}$ which is the same as Eq. (3.17). ■

Theorem 3.9 (Global Existence). *Let us now suppose $U = X$ and F satisfies the Lipschitz hypothesis in Eq. (3.14). Then there is a unique solution, $y(t)$ to the ODE in Eq. (3.12).*

Proof. We will use the standard method of Picard iterates. Namely, define $y_0(t) := y_0$ and then define $y_n(t)$ inductively by,

$$y_{n+1}(t) := x + \int_0^t F(\tau, y_n(\tau))\,dx(\tau). \tag{3.18}$$

Then from our assumptions and the definition of $y_n(t)$, we find for $n \geq 1$ that

$$
\begin{aligned}
\|y_{n+1}(t) - y_n(t)\| &= \left\| \int_0^t F(\tau, y_n(\tau))\, dx(\tau) - \int_0^t F(\tau, y_{n-1}(\tau))\, dx(\tau) \right\| \\
&\le \int_0^t \|F(\tau, y_n(\tau)) - F(\tau, y_{n-1}(\tau))\| \, \|dx(\tau)\| \\
&\le \int_0^t K(\tau) \|y_n(\tau) - y_{n-1}(\tau)\| \, \|dx(\tau)\| .
\end{aligned}
$$

Since,

$$
\|y_1(t) - y_0(t)\| = \left\| \int_0^t F(\tau, x)\, dx(\tau) \right\| \le \int_0^t \|F(\tau, x)\| \, \|dx(\tau)\| =: \delta(t) ,
$$

it follows by an application of Exercise 3.4 with

$$
\varepsilon_n(t) := \|y_{n+1}(t) - y_n(t)\|
$$

that

$$
\|y_{n+1}(t) - y_n(t)\| \le \int_0^t \|F(\tau, x)\| \, \|dx(\tau)\| \cdot \left(\int_0^t K(\tau) \|dx(\tau)\| \right)^n /n!. \quad (3.19)
$$

Since the right side of this equation is increasing in t, we may conclude by summing Eq. (3.19) that

$$
\sum_{n=0}^{\infty} \sup_{0 \le t \le T} \|y_{n+1}(t) - y_n(t)\| \le \left(\int_0^T \|F(\tau, x)\| \, dx(\tau) \right) e^{\int_0^T K(\tau)\|dx(\tau)\|} < \infty.
$$

Therefore, it follows that $y_n(t)$ is uniformly convergent on compact subsets of $[0, \infty)$ and therefore $y(t) := \lim_{n\to\infty} y_n(t)$ exists and is a continuous function. Moreover, we may now pass to the limit in Eq. (3.18) to learn this function y satisfies Eq. (3.12). Indeed,

$$
\begin{aligned}
&\left\| \int_0^t F(\tau, y_n(\tau))\, dx(\tau) - \int_0^t F(\tau, y(\tau))\, dx(\tau) \right\| \\
&\quad \le \int_0^t \|F(\tau, y_n(\tau)) - F(\tau, y(\tau))\| \, \|dx(\tau)\| \\
&\quad \le \int_0^t K(\tau) \|y_n(\tau) - y(\tau)\| \, \|dx(\tau)\| \\
&\quad \le \sup_{0 \le \tau \le t} \|y_n(\tau) - y(\tau)\| \cdot \int_0^t K(\tau) \|dx(\tau)\| \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

■

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