PATH INTEGRAL NOTES

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Abstract. These are some notes on heuristic and rigorous aspects of path integrals.

Contents

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1. Finite Dimensional Gaussian Integrals

Let A be a positive definite symmetric $N \times N$ matrix , then by a standard calculation,

$$Z_A := \int_{\mathbb{R}^N} e^{-\frac{1}{2}Ax\cdot x} dx = \sqrt{\det\left(\frac{2\pi}{A}\right)}. \tag{1.1}$$

Indeed we know that

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}|y|^2} dy = \left(\int_{\mathbb{R}} e^{-\frac{1}{2}y^2} dy\right)^N = (\sqrt{2\pi})^N$$

from which Eq. (1.1) follows after making the change of variables: $y = \sqrt{A}x$.

Notation 1.1. For $A > 0$, let μ_A be the probability measure on $\mathbb{R}^N$ defined by

$$d\mu_A(x) := \frac{1}{Z_A} e^{-\frac{1}{2}Ax\cdot x} dx = \sqrt{\det\frac{A}{2\pi}} e^{-\frac{1}{2}Ax\cdot x} dx. \tag{1.2}$$

Further let

$$L := L^A := \sum_{i,j=1}^N A^{-1}_{i,j} \partial_i \partial_j \text{ where } \partial_i := \frac{\partial}{\partial x_i}. \tag{1.3}$$

More invariantly, if we let $(v,w)_A := (Av,w)_{\mathbb{R}^N}$, then we may write $L = \sum_{j=1}^N \partial^2_{u_j}$ where $\{u_j\}_{j=1}^N$ is any orthonormal basis for $\left(\mathbb{R}^N, (\cdot,\cdot)_A\right)$.

Proposition 1.2 (Gaussian Integral formulas)**.** *The following Gaussian integral formulas hold;*

$$\int_{\mathbb{R}^N} e^{\lambda\cdot x} d\mu_A(x) = e^{\frac{1}{2}(A^{-1}\lambda\cdot\lambda)} \quad \forall \quad \lambda \in \mathbb{C}^N, \tag{1.4}$$

$$\int_{\mathbb{R}^N} f\left(x - \sqrt{t}y\right) d\mu_A(y) = \left(e^{t\frac{L}{2}} f\right)(x), \tag{1.5}$$

$$\int_{\mathbb{R}^N} f(x)\, d\mu_A(x) = \left(e^{\frac{L}{2}} f\right)(0), \tag{1.6}$$

and for $v \in \mathbb{R}^N$,

$$\int_{\mathbb{R}^N} (\partial_v f)(x)\, d\mu_A(x) = \int_{\mathbb{R}^N} (Av,x)\, f(x)\, d\mu_A(x) \tag{1.7}$$

or equivalently,

$$\int_{\mathbb{R}^N} (v,x)\, f(x)\, d\mu_A(x) = \int_{\mathbb{R}^N} \left(\partial_{A^{-1}v} f\right)(x)\, d\mu_A(x). \tag{1.8}$$

Applying this formula with f replaced by fg gives the integration by parts formula;

$$\int_{\mathbb{R}^N} \left(\partial_{A^{-1}v} f\right)(x)\, g(x)\, d\mu_A(x) = \int_{\mathbb{R}^N} f(x)\left(-\partial_{A^{-1}v} + (v,x)\right) g(x)\, d\mu_A(x) \tag{1.9}$$

which we may abbreviate as;

$$\partial^*_{A^{-1}v} = -\partial_{A^{-1}v} + M_{(v,x)}. \tag{1.10}$$

We also have the formula,

$$\int_{\mathbb{R}^N} p(x)\, d\mu_A(x) = \int_{\mathbb{R}^N} p(D_\lambda)\, e^{\lambda\cdot x}|_{\lambda=0} d\mu_A(x) = p(D_\lambda) \left.\int_{\mathbb{R}^N} e^{\lambda\cdot x} d\mu_A(x)\right|_{\lambda=0}$$
$$= p(D_\lambda)\, e^{\frac{1}{2}(A^{-1}\lambda\cdot\lambda)}. \tag{1.11}$$

Proof: Using the identity

$$Ax\cdot x - 2\lambda\cdot x = A(x - A^{-1}\lambda)\cdot(x - A^{-1}\lambda) - A^{-1}\lambda\cdot\lambda,$$

we find

$$\begin{aligned}\int_{\mathbb{R}^N} e^{\lambda\cdot x} d\mu_A(x) &= \frac{1}{Z}\int_{\mathbb{R}^N} e^{-\frac{1}{2}(Ax\cdot x - 2\lambda\cdot x)} dx \\ &= e^{\frac{1}{2}A^{-1}\lambda\cdot\lambda}\frac{1}{Z}\int_{\mathbb{R}^N} e^{-\frac{1}{2}A(x-A^{-1}\lambda)\cdot(x-A^{-1}\lambda)} dx \\ &= e^{\frac{1}{2}A^{-1}\lambda\cdot\lambda}\frac{1}{Z}\int_{\mathbb{R}^N} e^{-\frac{1}{2}Ax\cdot x} dx = e^{\frac{1}{2}A^{-1}\lambda\cdot\lambda}\end{aligned}$$

where in the third equality we used the translation invariance of Lebesgue measure. Eq. (1.4) now follows from the previous equation by analytic continuation.

Let us finish by showing that Eq. (1.5) and 1.4) are consistent. To this end, suppose that $f(x) = e^{\lambda\cdot x}$. Then on one hand;

$$\begin{aligned}\int_{\mathbb{R}^N} f\left(x - \sqrt{t}y\right) d\mu_A(y) &= \int_{\mathbb{R}^N} e^{\lambda\cdot\left(x-\sqrt{t}y\right)} d\mu_A(y) = e^{\lambda\cdot x}\int_{\mathbb{R}^N} e^{-\sqrt{t}\lambda\cdot y} d\mu_A(y) \\ &= e^{\lambda\cdot x} e^{\frac{t}{2}A^{-1}\lambda\cdot\lambda} = \exp\left(\lambda\cdot x + \frac{t}{2}A^{-1}\lambda\cdot\lambda\right)\end{aligned}$$

while on the other hand,

$$Le^{\lambda\cdot x} = A^{-1}_{ij}\lambda_i\lambda_j\cdot e^{\lambda\cdot x} = \left(A^{-1}\lambda\cdot\lambda\right) e^{\lambda\cdot x}$$

and therefore,

$$e^{tL/2} e^{\lambda\cdot x} = e^{\frac{t}{2}A^{-1}\lambda\cdot\lambda} e^{\lambda\cdot x}.$$

Q.E.D.

Remark 1.3 (Feynman diagrams interpretation 1)**.** Let $\{v_j\}_{j=1}^n \subset \mathbb{R}^N$. Using the integration by parts formula we find,

$$\begin{aligned}\int_{\mathbb{R}^N} \prod_{j=1}^n (v_j,x)\, d\mu_A(x) &= \int_{\mathbb{R}^N} \prod_{j=1}^{n-1} (v_j,x)(v_n,x)\, d\mu_A(x) \\ &= \int_{\mathbb{R}^N} \prod_{j=1}^{n-1} (v_j,x)\, \partial^*_{(A^{-1}v_n)} 1 d\mu_A(x) \\ &= \int_{\mathbb{R}^N} \partial_{(A^{-1}v_n)} \prod_{j=1}^{n-1} (v_j,x)\, d\mu_A(x) \\ &= \int_{\mathbb{R}^N} \sum_{k=1}^{n-1} \left(A^{-1}v_n, v_k\right) \prod_{,j\notin\{k,n\}} (v_j,x)\, d\mu_A(x).\end{aligned}$$

Continuing this way inductively one easily shows that to compute the integral, $\int_{\mathbb{R}^N}\prod_{j=1}^n (v_j,x)\, d\mu_A(x)$, we should:

1. put down n dots labeled by the $\{v_j\}_{j=1}^n$,
2. to each perfect pairing (n must be even else we get zero) of the v_j assign a weight which is the product $\left(v_i, A^{-1}v_j\right)$ over all pairs v_i and v_j which are paired, and
3. sum the result over all perfect pairings.

Definition 1.4. Given a polynomial, $p(x)$, on $\mathbb{R}^N$, the new polynomial, $e^{-L/2}p$, often denoted by $:p:$ is called the Wick ordered version of p.

Before going further let us point out that in much of what we do for a while, we may simplify the notation and take $A = I$ without any loss of generality. The following lemma explains why this is the case.

Lemma 1.5. *Let $(v,w)_A := Av\cdot w$ – an inner product on $\mathbb{R}^N$. Then for any orthonormal basis, $\{u_j\}_{j=1}^N$ of $\left(\mathbb{R}^N, (\cdot,\cdot)_A\right)$ we have*

$$\sum_{i,j=1}^N A^{-1}_{ij} e_i\otimes e_j = \sum_{k=1}^N u_k\otimes u_k \tag{1.12}$$

and

$$\sum_{i,j=1}^N A^{-1}_{ij}\partial_i\partial_j = \sum_{k=1}^N \partial^2_{u_k}. \tag{1.13}$$

Proof: Let $v, w \in \mathbb{R}^N$, then by contracting Eq. (1.12) with $(v,\cdot)_A \otimes (w,\cdot)_A$ we must show,

$$\sum_{i,j=1}^{N} A_{ij}^{-1} (v, e_i)_A (w, e_j)_A = \sum_{k=1}^{N} (v, u_k)_A (w, u_k)_A = (v, w)_A . \tag{1.14}$$

However,

$$(v, e_i)_A = v \cdot A e_i = \sum_{m=1}^{N} v \cdot e_m A_{im} \text{ and}$$
$$(w, e_j)_A = w \cdot A e_j = \sum_{n=1}^{N} w \cdot e_n A_{jn}$$

so the left side of Eq. (1.14) becomes,

$$\sum_{i,j=1}^{N} A_{ij}^{-1} (v, e_i)_A (w, e_j)_A = A_{ij}^{-1} A_{im} A_{jn} (v \cdot e_m)(w \cdot e_n) = \delta_{jm} A_{jn} (v \cdot e_m)(w \cdot e_n)$$
$$= A_{mn} (v \cdot e_m)(w \cdot e_n) = Av \cdot w = (v, w)_A$$

as desired. Eq. (1.13) follows from Eq. (1.12) by mapping $\mathbb{R}^N \otimes \mathbb{R}^N$ to the space of constant coefficient purely second order differential operators in such as way that $v \otimes w \to \partial_v \partial_w$. Q.E.D.

Proposition 1.6. *For all polynomials, p, and q,*

$$\mu_A (: p : \cdot : q :) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} (\partial_{i_1} \dots \partial_{i_n} p)(0) (\partial_{j_1} \dots \partial_{j_n} q)(0) \tag{1.15}$$

and

$$\mu_A (p \cdot q) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} \mu_A (\partial_{i_1} \dots \partial_{i_n} p) \cdot \mu_A (\partial_{j_1} \dots \partial_{j_n} q) . \tag{1.16}$$

Proof: Let

$$P(t,x) := \left(e^{-tL/2} p\right)(x), \ Q(t,x) = \left(e^{-tL/2} q\right)(x), \text{ and}$$
$$h(t) := e^{tL/2} (P(t) \cdot Q(t)) \text{ – where } x \text{ has been suppressed.}$$

Then by a simple exercise in the chain and product rule we find

$$\dot{h}(t) = \sum_{i,j} A_{ij}^{-1} e^{tL/2} (\partial_i P(t) \cdot \partial_j Q(t)) .$$

Then working inductively it follows that

$$h^{(n)}(t) = \sum_{i_1, j_1, \dots, i_n, j_n} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} (\partial_{i_1} \dots \partial_{i_n} P(t)) (\partial_{j_1} \dots \partial_{j_n} Q(t))$$

and in particular that,

$$h^{(n)}(0) = \sum_{i_1, j_1, \dots, i_n, j_n} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} (\partial_{i_1} \dots \partial_{i_n} p) (\partial_{j_1} \dots \partial_{j_n} q) .$$

Since h is a polynomial in t (hence analytic), we have

$$e^{L/2} \left(e^{-L/2} p \cdot e^{-L/2} q\right) = e^{L/2} (P(1) \cdot Q(1)) = h(1) = \sum_{n=0}^{\infty} \frac{1}{n!} h^{(n)}(0)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} (\partial_{i_1} \dots \partial_{i_n} p) (\partial_{j_1} \dots \partial_{j_n} q) . \tag{1.17}$$

Evaluating this expression at $x = 0$ gives Eq. (1.15). Replacing p and q by $e^{L/2} p$ and $e^{L/2} q$ respectively in Eq. (1.17) and then evaluating at $x = 0$ gives Eq. (1.16). Q.E.D.

Corollary 1.7. *Let $S(\mathbb{R}^N)$ denote the symmetric tensor algebra over $\mathbb{R}^N$ which we identify with the space of polynomial functions on $\mathbb{R}^N$. On this space, let*

$$\langle p, q \rangle_A = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1, j_1, \dots, i_n, j_n = 1}^{\infty} A_{i_i, j_1}^{-1} \dots A_{i_n, j_n}^{-1} \left(\partial_{i_1} \dots \partial_{i_n} e^{-L/2} p\right)(0) \left(\partial_{j_1} \dots \partial_{j_n} e^{-L/2} q\right)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{u_1, \dots, u_n \in S} \left(\partial_{u_1} \dots \partial_{u_n} e^{-L/2} p\right) \left(\partial_{u_1} \dots \partial_{u_n} e^{-L/2} q\right)|_{x=0}$$

where S is an orthonormal basis for $\left(\mathbb{R}^N, (\cdot,\cdot)_A\right)$. Then the (Fock-Kakutani-Ito – Segal-Bargmann) map,

$$\left(S(\mathbb{R}^N), (\cdot,\cdot)_{L^2(\mu_A)}\right) \ni p \to e^{-L/2} p \in \left(S(\mathbb{R}^N), \langle \cdot,\cdot \rangle_A\right)$$

is an isometric isomorphism of vector spaces. By completing the right side of this equation and using continuity, we get a unitary map between Hilbert spaces;

$$L^2(\mu_A) \ni p \to \text{“}e^{-L/2} p\text{”} \in \left(\overline{S(\mathbb{R}^N)}, \langle \cdot,\cdot \rangle_A\right).$$

The right hand side is called the ***Bosonic Fock space*** *over $\mathbb{R}^N$.*

Remark 1.8. Let us consider the free Euclidean field, i.e. the Gaussian measure is informally given by;

$$d\mu(\varphi) = \frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{R}^d} \left[|\nabla \varphi(x)|^2 + m^2 \varphi^2(x)\right] dx\right) \mathcal{D}\varphi \tag{1.18}$$

$$= \frac{1}{Z} \exp\left(-\frac{1}{2}\left(\left(-\Delta + m^2\right)\varphi, \varphi\right)_{L^2(\mathbb{R}^d, dx)}\right)\mathcal{D}\varphi,$$

which we view as a measure on $L^2\left(\mathbb{R}^d, dx\right)$. Working as above if $f, g \in L^2\left(\mathbb{R}^d\right)$, then we should have

$$\int_{L^2(\mathbb{R}^d, dx)} (\varphi, f)(\varphi, g)\, d\mu(\varphi) = \left(\left(-\Delta + m^2\right)^{-1} f, g\right). \tag{1.19}$$

Taking f and g, formally, to be δ_x and δ_y respectively, the above formula leads to

$$\int_{L^2(\mathbb{R}^d, dx)} \varphi(x)\varphi(y)\, d\mu(\varphi) = \left(\left(-\Delta + m^2\right)^{-1}\delta_x, \delta_y\right) =: \Delta_m(x - y),$$

where Δ_m is called the Euclidean propagator. This propagator satisfies:

(1) If $d = 1$ then

$$\Delta_m(x) = \frac{1}{2m} e^{-m|x|}.$$

(2) For $d \geq 2$,

$$\Delta_m(x) \backsim \begin{cases} -\ln|x| & \text{if} \quad d = 2 \\ |x|^{d-2} & \text{if} \quad d \geq 3 \end{cases} \quad \text{for } |x| \ll 1$$

and $\Delta_m(x) \backsim e^{-m|x|}$ for $|x| \ggg 1$. In particular for $d \geq 2$, we have

$$\int_{L^2(\mathbb{R}^d, dx)} \varphi^2(x)\, d\mu(\varphi) = \infty$$

from which we might expect that there is no Gaussian measure, μ, on $L^2\left(\mathbb{R}^d, dx\right)$ (or any class of functions for that matter) such that Eq. (1.19) holds.

Lemma 1.9. *For any polynomials, $p(x)$ and $q(x)$, on $\mathbb{R}^N$ we have*

$$\mu_A(:(v, x)p(x): \cdot q(x)) = \mu_A(:p(x): \partial_{A^{-1}v} q(x)) \tag{1.20}$$

where

$$\mu_A(f) = \mu_A(f(x)) := \int_{\mathbb{R}^N} f(x)\, d\mu_A(x).$$

Proof: Since

$$\frac{d}{dt}\left(e^{-tL/2} M_{(v,x)} e^{tL/2}\right) = -e^{-tL/2}\left[\frac{L}{2}, M_{(v,x)}\right] e^{tL/2}$$
$$= -e^{-tL/2} A^{-1}_{ij} (v, e_i)\, \partial_{e_i} e^{tL/2} = -\partial_{A^{-1}v},$$

it follows that

$$e^{-tL/2} M_{(v,x)} e^{tL/2} = M_{(v,x)} - t\partial_{A^{-1}v} \tag{1.21}$$

and hence

$$e^{-L/2} M_{(v,x)} = \left(M_{(v,x)} - t\partial_{A^{-1}v}\right) e^{-L/2}. \tag{1.22}$$

Therefore,

$$:(v, x)p(x) := (v, x): p(x): -\partial_{A^{-1}v}: p(x) := \partial^*_{A^{-1}v}: p(x):$$

and hence

$$\mu_A(:(v, x)p(x): \cdot q(x)) = \mu_A\left(\partial^*_{A^{-1}v}: p(x): \cdot q(x)\right) = \mu_A\left(:p(x): \partial_{A^{-1}v} q(x)\right).$$

Q.E.D.

Remark 1.10 (Feynman Interpretation 2)**.** Let us now work out the Feynman rules for computing an integral of the form,

$$J((v_1, m_1), \ldots, (v_n, m_n)) := \int_{\mathbb{R}^N} \prod_{j=1}^n :(v_j, x)^{m_j}: d\mu_A(x).$$

Using the integration by parts formula in Eq. (1.20) we have,

$$\int_{\mathbb{R}^N} \prod_{j=1}^n :(v_j, x)^{m_j}: \cdot :(v_n, x)^{m_n - 1}(v_n, x): d\mu_A(x)$$
$$= \int_{\mathbb{R}^N} \left(\partial_{A^{-1}v_n} \prod_{j=1}^n :(v_j, x)^{m_j}:\right) :(v_n, x)^{m_n - 1}: d\mu_A(x),$$

where

$$\partial_{A^{-1}v_n} \prod_{j=1}^n :(v_j, x)^{m_j} = \sum_{k=1}^{n-1} \prod_{j \neq k, n} :(v_j, x)^{m_j}: \cdot :(v_n, x)^{m_n - 1}: \cdot \partial_{A^{-1}v_n}:(v_k, x)^{m_k}:$$
$$= \sum_{k=1}^{n-1} \prod_{j \neq k, n} :(v_j, x)^{m_j}: \cdot :(v_n, x)^{m_n - 1}: \cdot m_k\left(A^{-1}v_n, v_k\right):(v_k, x)^{m_k - 1}:.$$

Using these formula inductively and a little thought leads to the following Feynman rules for computing $J((v_1, m_1), \ldots, (v_n, m_n))$:

(1) Draw n dots labeled by $\{v_j\}_{j=1}^n$ and to the j^{th} – dot, attach m_j legs.
(2) Consider all possible pairings of the legs (iff possible) with no self connections, i.e. no pairings of a leg of some v_i with another leg of v_i.
(3) Given a pairing graph as above associate the weight which is the product over terms of the form $\left(A^{-1}v_i, v_j\right)$ – one for each leg which connects v_i to v_j.
(4) Sum the above weights over all allowed pairing graphs.

2. Infinite Dimensional Gaussian Integrals

For general references on Gaussian measures in infinite dimensions, see [1, 2, 6].

It will now be convenient to get rid of the matrix A in the above formula. We do this by letting $(v, w) := (v, w)_A := v \cdot Aw$. With this notation, a small exercise in linear algebra shows that Eq. (1.4) may be written in this notation as

$$\int_{\mathbb{R}^N} e^{\lambda(x)} d\mu_A(x) = e^{\frac{1}{2}(\lambda,\lambda)_*} \quad \forall \quad \lambda \in \left(\mathbb{R}^N\right)^*,$$

where $(\cdot,\cdot)_*$ is the inner product on $\left(\mathbb{R}^N\right)^*$ dual to $(\cdot,\cdot)_A$. With this as motivation we now proceed to the infinite dimensional setting.

Given a real separable Hilbert space H, we would like to understand,

$$d\mu(x) := \frac{1}{Z} e^{-\frac{1}{2}(x,x)_H} \mathcal{D}x \tag{2.1}$$

as a Gaussian measure on H. A formal definition would be that μ is to be the unique measure on H such that

$$\int_H e^{\lambda(x)} d\mu(x) = e^{\frac{1}{2}(\lambda,\lambda)_{H^*}} \text{ for all } \lambda \in H^*. \tag{2.2}$$

Let us now suppose that we have chosen an orthonormal basis $\{e_j\}_{j=1}^\infty$ for H and let us identify H with ℓ^2 via, $x \to \{x_j = (x,e_j)\}_{j=1}^\infty$. In this notation, Eqs. (2.1) and (2.2) become,

$$d\mu(x) := \frac{1}{Z} e^{-\frac{1}{2}(x,x)_{\ell^2}} \mathcal{D}x, \tag{2.3}$$

where $(x,x)_{\ell^2} := \sum_{j=1}^\infty x_j^2$, $\mathcal{D}x = \prod_{i=1}^{\infty} dx_i$, and

$$\int_{\ell^2} e^{(\lambda,x)} d\mu(x) = e^{\frac{1}{2}(\lambda,\lambda)_{\ell^2}} \text{ for all } \lambda \in \mathcal{F} \tag{2.4}$$

where $\mathcal{F} \subset \ell^2$ is the collection of sequences with only finitely many non-zero terms.

In this case, the Gaussian measure that we are trying to construct is formally given by the expression,

$$\begin{aligned} d\mu(x) &= \frac{1}{(\sqrt{2\pi})^\infty} \exp\left(-\frac{1}{2}\sum_{i=1}^\infty x_i^2\right) \prod_{i=1}^\infty dx_i \\ &= \prod_{i=1}^\infty \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x_i^2}\, dx_i\right) =: \prod_{i=1}^\infty p_1(dx_i), \end{aligned}$$

where $p_1(dy) := \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy\right)$.

This suggests that we define $\mu = p_1^{\otimes\mathbb{N}}$, the infinite product measure on $\mathbb{R}^{\mathbb{N}}$.

Theorem 2.1. *Let $\mu = p_1^{\otimes\mathbb{N}}$ be the infinite product measure on $\left(\mathbb{R}^{\mathbb{N}}, \mathcal{B} := \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}\right)$ where $\mu = p_1^{\otimes\mathbb{N}}$ as described above. For $a = (a_1, a_2, \dots) \in (0,\infty)^{\mathbb{N}}$, define*

$$X_a = \ell^2(a) = \{x \in \mathbb{R}^{\mathbb{N}} : \sqrt{\sum_{i=1}^\infty a_i x_i^2} =: \|x\|_a < \infty\},$$

i.e. $X_a = L^2(\mathbb{N}, a)$ where a now denotes the measure on $\mathbb{N}$ determined by $a(\{i\}) = a_i$ for all $i \in \mathbb{N}$. Then $X_a \in \mathcal{B}$, $\mathcal{B}_{X_a} := \{A \cap X_a : A \in \mathcal{B}\}$ ($\mathcal{B}_{X_a}$ is the Borel σ – field on X_a) and

$$\mu(X_a) = \begin{cases} 1 & \text{if } \sum_{i=1}^\infty a_i < \infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

Assuming that $\sum_{i=1}^\infty a_i < \infty$, $\mu_a := \mu|_{\mathcal{B}_{X_a}}$ is a the unique probability measure on $(X_a, \mathcal{B}_{X_a})$ which satisfies

$$\int_{X_a} f(x_1,\dots,x_n) d\mu_a(x) = \int_{\mathbb{R}^n} f(x_1,\dots,x_n) p_1(dx_i)\dots p_1(dx_n) \tag{2.6}$$

for all bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and $n = 1,2,3,\dots$. Moreover, if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n, 0, 0, \dots)$, then

$$\int_{X_a} e^{(\lambda,x)_{\ell^2}} d\mu(x) = e^{\frac{1}{2}(\lambda,\lambda)_{\ell^2}}. \tag{2.7}$$

Proof: For $N \in \mathbb{N}$, let $q_N : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be defined by $q_N(x) = \sum_{i=1}^N a_i x_i^2$.[1] Then for any $\varepsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu &= \int_{\mathbb{R}^{\mathbb{N}}} \lim_{N\to\infty} e^{-\varepsilon q_N/2} d\mu \overset{\text{M.C.T.}}{=} \lim_{N\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q_N/2} d\mu \\ &= \lim_{N\to\infty} \int_{\mathbb{R}^N} e^{-\frac{\varepsilon}{2}\sum_1^N a_i x_i^2} \prod_{i=1}^N p_1(dx_i) \end{aligned}$$

[1]**Technicalities:** It is easily seen that q_N is $\mathcal{B}$ – measurable. Therefore, $q := \sup_{N\in\mathbb{N}} q_N$ (also notice that $q_N \uparrow q$ as $N \to \infty$) is $\mathcal{B}$ – measurable as well and hence

$$X_a = \{x \in \mathbb{R}^{\mathbb{N}} : q(x) < \infty\} \in \mathcal{B}.$$

Similarly, if $x_0 \in X_a$, then $q(\cdot - x_0) = \sup_{N\in\mathbb{N}} q_N(\cdot - x_0)$ is $\mathcal{B}$ – measurable and therefore for $r > 0$,

$$B(x_0,r) = \{x \in X_a : \|x - x_0\|_a < r\} = \{x \in \mathbb{R}^{\mathbb{N}} : q(\cdot - x_0) < r^2\} \in \mathcal{B}$$

which shows that $\mathcal{B}_{X_a} \subset \mathcal{B}$ and hence $\mathcal{B}_{X_a} \subset \{A \cap X_a : A \in \mathcal{B}\}$. To prove the reverse inclusion, let $i : X_a \to \mathbb{R}^{\mathbb{N}}$ be the inclusion map and recall that

$$\begin{aligned} \{A \cap X_a : A \in \mathcal{B}\} &= i^{-1}(\mathcal{B}) = i^{-1}\left(\sigma\left(\pi_j^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_{\mathbb{R}}\right)\right) \\ &= \sigma\left(i^{-1}\pi_j^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_{\mathbb{R}}\right) \\ &= \sigma\left((\pi_j \circ i)^{-1}(B) : j \in \mathbb{N} \text{ and } B \in \mathcal{B}_{\mathbb{R}}\right) = \sigma(\pi_j \circ i : j \in \mathbb{N}). \end{aligned}$$

Since $\pi_j \circ i \in X_a^*$ for all j, we see from this expression that

$$\{A \cap X_a : A \in \mathcal{B}\} \subset \sigma(X_a^*) \subset \mathcal{B}_{X_a} \subset \{A \cap X_a : A \in \mathcal{B}\}.$$

$$= \lim_{N\to\infty} \prod_{i=1}^{N} \int_{\mathbb{R}} e^{-\frac{\varepsilon}{2}a_i x^2} p_1(dx). \tag{2.8}$$

$$\{A \cap X_a : A \in \mathcal{B}\} \subset \sigma(X_a^*) \subset \mathcal{B}_{X_a} \subset \{A \cap X_a : A \in \mathcal{B}\}.$$

Using

$$\int e^{-\frac{\lambda}{2}x^2} p_1(x)dx = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{\lambda+1}{2}x^2} dx = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2\pi}{\lambda+1}} = \frac{1}{\sqrt{\lambda+1}}$$

in Eq. (2.8) we learn that

$$\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu = \lim_{N\to\infty} \prod_{1}^{N} \frac{1}{\sqrt{1+\varepsilon a_i}} = \sqrt{\lim_{N\to\infty} \prod_{1}^{N} (1+\varepsilon a_i)^{-1}}$$

or equivalently that

$$-\log\left(\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu\right) = \frac{1}{2} \sum_{i=1}^{\infty} \ln(1+\varepsilon a_i). \tag{2.9}$$

Notice that there is $\delta > 0$ such that

$$\ln(1+x) \le x \ \forall x \ge 0 \text{ and } \ln(1+x) \ge x/2 \text{ for } x \in [0,\delta]. \tag{2.10}$$

If $\limsup_{i\to\infty} a_i \ne 0$, then $\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) = \infty$ for all $\varepsilon > 0$. If $\lim_{i\to\infty} a_i = 0$ but $\sum_{i=1}^{\infty} a_i = \infty$, then using Eq. (2.10) $\ln(1+\varepsilon a_i) \ge \varepsilon a_i/2$ for all i large and hence again $\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) = \infty$. If $\sum_{i=1}^{\infty} a_i < \infty$ then by Eq. (2.10),

$$\sum_{i=1}^{\infty} \ln(1+\varepsilon a_i) \le \varepsilon \sum_{i=1}^{\infty} a_i \to 0 \text{ as } \varepsilon \downarrow 0.$$

In summary,

$$-\lim_{\varepsilon\downarrow 0} \log\left(\int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu\right) = \begin{cases} \infty & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 0 & \text{if } \sum_{i=1}^{\infty} a_i < \infty \end{cases}$$

or equivalently,

$$\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^{\mathbb{N}}} e^{-\varepsilon q/2} d\mu = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 1 & \text{if } \sum_{i=1}^{\infty} a_i < \infty. \end{cases}$$

Since $e^{-\varepsilon q/2} \le 1$ and $\lim_{\varepsilon\downarrow 0} e^{-\varepsilon q/2} = 1_{X_a}$, the previous equation along with the dominated convergence theorem shows that

$$\mu(X_a) = \int_{\mathbb{R}^{\mathbb{N}}} 1_{X_a} d\mu = \begin{cases} 0 & \text{if } \sum_{i=1}^{\infty} a_i = \infty \\ 1 & \text{if } \sum_{i=1}^{\infty} a_i < \infty. \end{cases}$$

proving Eq. (2.5).

Finally Eq. (2.6) follows from the definition of μ and the fact that

$$\int_{X_a} f(x_1,\ldots,x_n) d\mu_a(x) = \int_{\mathbb{R}^{\mathbb{N}}} f(x_1,\ldots,x_n) d\mu(x).$$

By the definition of infinite product measure, the integral on the left side of Eq. (2.7) may be reduced to a finite dimensional Gaussian integral. The resulting integral is easily computed to give the right side Eq. (2.7). Q.E.D.

Theorem 2.2. *Suppose that H and K are separable Hilbert spaces, H is a dense subspace of K, and the inclusion map, $i : H \to K$ is continuous. Then there exists a Gaussian measure, ν, on K such that*

$$\int_K e^{\lambda(x)} d\nu(x) = \exp\left(\frac{1}{2}(\lambda,\lambda)_{H^*}\right) \text{ for all } \lambda \in K^* \subset H^* \tag{2.11}$$

iff $i : H \to K$ is Hilbert Schmidt. Recalling the Hilbert Schmidt norm of i and its adjoint, i^*, are the same, the following conditions are equivalent;

1. *$i : H \to K$ is Hilbert Schmidt,*
2. *$i^* : K \to H$ is Hilbert Schmidt,*
3. $\mathrm{tr}(i\,i^*) < \infty$
4. $\mathrm{tr}(i^* i) < \infty$.

Proof: Let us start with the direction that we are most interested in. Namely, if $i : H \to K$ is Hilbert Schmidt, then there exists a measure ν on K such that Eq. (2.11) holds. In this case, the operator, $A := i^* i : H \to H$, is a self-adjoint trace class operator and hence by the spectral theorem, there exists an orthonormal basis, $\{e_j\}_{j=1}^{\infty}$ for H such that $Ae_j = a_j e_j$ with $a_j > 0$ and $\sum_{j=1}^{\infty} a_j < \infty$. Observe that

$$(e_j, e_k)_K = (ie_j, ie_k)_K = (i^* i e_j, e_k)_H = (Ae_j, e_k)_H = a_j \delta_{jk}$$

and therefore, $\left\{a_j^{-1/2} e_j\right\}_{j=1}^{\infty}$ is an orthonormal basis for K. Hence the map, $U : X(a) \to K$ defined by $Ux = \sum_{j=1}^{\infty} x_j e_j$ is unitary and so is $U|_{\ell^2} : \ell^2 \to H$. (Notice that

$$\|Ux\|_K^2 = \sum_{j=1}^{\infty} x_j^2 \|e_j\|_K^2 = \sum_{j=1}^{\infty} x_j^2 a_j.)$$

We may now define $v := U_*\mu := \mu \circ U^{-1}$ – a probability measure on K. Moreover, if $\lambda \in K^*$, then

$$\int_K e^{\lambda(k)} d\nu(k) = \int_K e^{\lambda(Ux)} d\mu(x) = \exp\left(\frac{1}{2}(\lambda\circ U, \lambda\circ U)_{(\ell^2)^*}\right) = \exp\left(\frac{1}{2}(\lambda,\lambda)_{H^*}\right).$$

We will sketch two proofs of the converse direction – each of which makes use of a result (see [1, 2, 6]) which will not be proved here. In each case we start with H densely contained and continuous embedded in K and assume there exists a measure ν on K satisfying Eq. (2.11).

1. For the first proof, we use the fact that under the above assumptions, $i : H \to K$, is compact. Therefore $A = i^* i$ is a compact self-adjoint operator. Therefore there exists an orthonormal basis, $\{e_j\}_{j=1}^\infty$ for H such that $Ae_j = a_j e_j$ with $a_j > 0$ and $\lim_{j\to\infty} a_j = 0$. Let $P_N : K \to \mathrm{span}\,(e_1, \dots, e_N)$ be defined by $P_N(k) = \sum_{j=1}^N (k, e_j)_H e_j$. Then, working as above, $(e_j, e_k)_K = a_j \delta_{jk}$ so that

$$\|P_N(k)\|_K^2 = \sum_{j=1}^N a_j (k, e_j)_H^2 \uparrow \|k\|_K^2 .$$

Therefore by the monotone convergence theorem,

$$\begin{aligned} \int_K \|k\|_K^2 \, d\nu(k) &= \lim_{N\to\infty} \sum_{j=1}^N a_j \int_K (k, e_j)_H^2 \, d\nu(k) \\ &= \lim_{N\to\infty} \sum_{j=1}^N a_j = \sum_{j=1}^\infty a_j = \sum_{j=1}^\infty \|e_j\|_K^2 . \end{aligned}$$

However, by Fernique's or Skorohod's theorem, we know that $\int_K \|k\|_K^2 \, d\nu(k) < \infty$.

2. It is known that for any orthonormal basis $\{e_j\}_{j=1}^\infty$ of H which is contained in

$$H_* = \{h \in H : (h, \cdot)_H \text{ extends to an element of } K^*\}$$

has the property that $P_N(k) = \sum_{j=1}^N (k, e_j)_H e_j \to k$ in $L^p(\nu)$ for all $p < \infty$. Therefore,

$$\begin{aligned} \int_K \|k\|_K^2 \, d\nu(k) &= \lim_{N\to\infty} \int_K \|P_N(k)\|_K^2 \, d\nu(k) \\ &= \lim_{N\to\infty} \int_K \left\| \sum_{j=1}^N (k, e_j)_H e_j \right\|_K^2 d\nu(k) \\ &= \lim_{N\to\infty} \int_K \sum_{j,l=1}^N (k, e_j)_H (k, e_l)_H (e_j, e_k)_K \, d\nu(k) \\ &= \lim_{N\to\infty} \sum_{j,l=1}^N \delta_{jk} (e_j, e_k)_K = \sum_{j=1}^\infty \|e_j\|_K^2 . \end{aligned}$$

So again we may now apply Fernique's theorem to finish the proof. Q.E.D.

Lemma 2.3. *If H is a Hilbert space and $A : H \to H$ is a positive trace class operator. We may define $(x, y)_A := (Ax, y)$ for all $x, y \in H$. Then let K denote the completion of H in the norm, $\|\cdot\|_A := \sqrt{(\cdot,\cdot)_A}$. Then the inclusion map, $i : H \to K$ will be Hilbert Schmidt and hence K will support a Gaussian measure with variance determined by H.*

Remark 2.4. Suppose that $H \subset K$ and ν is a Gaussian measure on K such that Eq. (2.11) holds and $\{e_j\}_{j=1}^\infty$ is any orthonormal basis for H. Then for $k_1, k_2 \in K$, we have

$$\begin{aligned} \int_K (k_1, k)_K (k_2, k)_K \, d\nu(k) &= ((k_1, \cdot)_K (k_2, \cdot)_K)_{H^*} = \sum_{j=1}^\infty (k_1, e_j)_K (k_2, e_j)_K \\ &= \sum_{j=1}^\infty (k_1, ie_j)_K (k_2, ie_j)_K = \sum_{j=1}^\infty (i^* k_1, e_j)_H (i^* k_2, e_j)_H \\ &= (i^* k_1, i^* k_2)_H = (k_1, ii^* k_2)_K = (k_1, i^* k_2)_K \end{aligned}$$

Example 2.5 (Wiener measure)**.** Let H denote the set of functions $h : [0, T] \to \mathbb{R}^d$ which are absolutely continuous and satisfy $h(0) = 0$ and $(h, h)_H = \int_0^1 |h'(s)|^2 ds < \infty$. The informal expression for Wiener measure is then given by

$$d\mu(\omega) = \frac{1}{Z} \exp\left(-\frac{1}{2} \int_0^T |\dot\omega(t)|^2 \, dt \right) \mathcal{D}\omega \text{ for } \omega \in H.$$

We are going to show that the measure μ may be constructed on $L^2([0, T], \mathbb{R}^d)$. Before doing this let us compute i^* where $i : H \to L^2$ is the inclusion operator. To this end, let $h \in H$ and $f \in L^2$, then

$$\begin{aligned} (ih, f)_{L^2} &= \int_0^T h(t) \cdot f(t) \, dt = \int_0^T \left(\int_0^t h'(s) \, ds \right) \cdot f(t) \, dt \\ &= \int_{[0,T]^2} 1_{0\le s\le t} h'(s) \cdot f(t) \, dt ds = \int_0^T h'(s) \cdot \left(\int_s^T f(t) \, dt \right) ds \\ &= (h, i^* f)_H , \end{aligned}$$

where

$$\begin{aligned} (i^* f)(\tau) &= \int_0^\tau \left(\int_s^T f(t) \, dt \right) ds = \int_{[0,T]^2} 1_{0\le s\le \tau} \cdot 1_{s\le t\le T} f(t) \, dt ds \\ &= \int_0^T \min(t, \tau) f(\tau) \, d\tau. \end{aligned}$$

From this we expect that

$$\mathrm{tr}\,(i\, i^*) = d \cdot \int_0^T \min(t, t) \, dt = d \cdot T^2/2$$

which is finite. To see this is correct use Corollary 7.2 below to conclude

$$\sum_{n=1}^{\infty}\int_0^T |h_n(t)|^2\,dt = d\int_0^T t\,dt = d\cdot T^2/2.$$

It now follows from Remark 2.4 that

$$\int_{L^2}(f,\omega)_{L^2}(g,\omega)_{L^2}\,d\mu(\omega) = (f,i^*g)_{L^2} = \int_{[0,T]^2} f(s)\,g(t)\min(s,t)\,dsdt.$$

In particular taking $f=\delta_s$ and $g=\delta_t$ leads us to expect that

$$\int_{L^2}\omega(s)\otimes\omega(t)\,d\mu(\omega) = \min(s,t)\sum_{i=1}^d e_i\otimes e_i$$

is a meaningful computation. In fact, Wiener [8, 7] proved in 1923 that $\mu(W_T)=1$ where

$$W_T := \left\{\omega\in C([0,T]\to\mathbb{R}^d) : \omega(0)=0\right\}.$$

Example 2.6. Now consider the measure $d\mu(\varphi)$ of Remark 1.8. In this case one can show that the measure never lives on $L^2(\mathbb{R}^d)$ no matter the dimension. To see this let H be the Sobolev space of one derivative in L^2. In this case we have $i^* = \left(-\Delta+m^2\right)^{-1}$. Indeed, if $u := i^*g$, then

$$(f,g)_{L^2} = (if,g)_{L^2} = (f,u)_H = \int_{\mathbb{R}^d}\left(\nabla f\cdot\nabla u + m^2 f\cdot u\right)dx\ \forall\ f\in H,$$

which, by Elliptic regularity or by the Fourier transform, implies the distribution, Δu, is an L^2 – function and

$$(f,g)_{L^2} = \left(f,\left(-\Delta+m^2\right)u\right).$$

Hence we have $\left(-\Delta+m^2\right)u = g$ or $i^*g = u = \left(-\Delta+m^2\right)^{-1}g$ as claimed.

Informally, now,

$$\operatorname{tr}(i\,i^*) = \int_{\mathbb{R}^d}\left(-\Delta+m^2\right)^{-1}(x,x)\,dx = \infty.$$

because when $d\geq 2$, $\left(-\Delta+m^2\right)^{-1}(x,x)=\infty$ and or $d=1$ $\left(-\Delta+m^2\right)^{-1}(x,x) = c>0$ for all $x\in\mathbb{R}$. To give a rigorous proof, notice that $i\,i^*$ is unitarily equivalent to the the multiplication operator, $\left(k^2+m^2\right)^{-1}$ which has continuous spectrum, it follows that $\left(-\Delta+m^2\right)^{-1}$ is not compact let alone trace class. In general on non-atomic spaces with no infinite atoms any non-zero multiplication operator is not trace class. Indeed, suppose M_f is the multiplication operator on $L^2(X,m)$ and observe that we may assume $f\geq 0$ since $|M_f| = M_{|f|}$. Then for some $\varepsilon>0$ we will have $\mu(f\geq\varepsilon)>0$. Since m is non-atomic, we may write $\{f\geq\varepsilon\}$ as a disjoint union of $\{A_j\}$ where $\infty > m(A_j) > 0$. Then the functions, $\left\{\frac{1}{\sqrt{m(A_j)}}1_{A_j}\right\}_{j=1}^{\infty}$ forms an orthonormal subset of $L^2(m)$ and therefore,

$$\begin{aligned}\operatorname{tr}(M_f) &\geq \sum_{j=1}^{\infty}\left(\frac{1}{\sqrt{m(A_j)}}1_{A_j}, f\frac{1}{\sqrt{m(A_j)}}1_{A_j}\right) = \sum_{j=1}^{\infty}\frac{1}{m(A_j)}\int_{A_j} f\,dm \\ &\geq \sum_{j=1}^{\infty}\frac{1}{m(A_j)}\int_{A_j}\varepsilon\,dm = \varepsilon\cdot\infty = \infty.\end{aligned}$$

Example 2.7. Part of the problem above was the non-compactness of $\mathbb{R}^d$. To avoid this issue, let us replace $\mathbb{R}^d$ be a $\mathbb{T}^d$ – the d – dimensional torus which we identify with $[0,2\pi]^d/\backsim$ where $\backsim$ is the usual identification of the endpoints. We will denote points in $[0,2\pi]^d$ by θ and let $d\theta$ denote normalized Haar measure on $\mathbb{T}^d$. In this case $\left\{\frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2+m^2}}\right\}_{n\in\mathbb{Z}^d}$ is an orthonormal basis for $H(\mathbb{T}^d)$ equipped with the inner product;

$$(f,g)_{H(\mathbb{T}^d)} := \int_{\mathbb{T}^d}\left[\nabla f(\theta)\cdot\nabla g(\theta) + m^2 f(\theta)\,g(\theta)\right]d\theta.$$

Hence if $i : H(\mathbb{T}^d)\to L^2(\mathbb{T}^d)$ is the inclusion map, then

$$\|i\|_{H.S.}^2 = \sum_{n\in\mathbb{Z}^d}\left\|\frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2+m^2}}\right\|_{L^2(\mathbb{T}^d)}^2 = \sum_{n\in\mathbb{Z}^d}\frac{1}{\|n\|^2+m^2}$$

which is finite iff $d=1$. When $d=2$ the sum is logarithmically divergent and it is worse when $d=3$. This can also be understood by noting that $\left(-\Delta+m^2\right)^{-1}(\theta-\alpha)$ has the same singularity structure as Example 2.6 above.

Hence we need to take K even bigger than $L^2(\mathbb{T}^d)$ – however only just barely when $d=2$. For example for any $s\in\mathbb{R}$ and $f\in C^\infty(\mathbb{T}^2)$,

$$\|f\|_s^2 := \sum_{n\in\mathbb{Z}^2}\left(\|n\|^2+m^2\right)^s\left|\hat f(n)\right|^2$$

where $\hat f(n) := \int_{\mathbb{T}^2} f(\theta)\,e^{-in\cdot\theta}d\theta$. So $\|f\|_0^2 = \|f\|_{L^2(\mathbb{T}^2)}^2$, $\|f\|_1^2 = \|f\|_H$ and for any $s<0$, $\|f\|_s^2\leq\|f\|_0^2$. Let K_s denote the completion of $C^\infty(\mathbb{T}^2)$ in the s – norm. This is the Sobolev space of s – derivatives in L^2. For any $s<0$, the inclusion map, $i : H = K_1\to K_s$ is Hilbert Schmidt. Indeed, we now have

$$\|i\|_{H.S.}^2 = \sum_{n\in\mathbb{Z}^2}\left\|\frac{e^{in\cdot\theta}}{\sqrt{\|n\|^2+m^2}}\right\|_{K_s}^2 = \sum_{n\in\mathbb{Z}^2}\frac{1}{\|n\|^2+m^2}\left(\|n\|^2+m^2\right)^s$$

$$= \sum_{n \in \mathbb{Z}^2} \frac{1}{\left(\|n\|^2 + m^2\right)^{1+|s|}} < \infty.$$

3. Semi-Group Property and the Operator Connection

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a "nice" potential function and for $x, y \in \mathbb{R}^d$ and $T > 0$, let

$$H(x,y,T) := \left\{ \omega : [0,T] \to \mathbb{R}^d : \omega(0) = x,\ \omega(T) = y \text{ and } \int_0^T |\dot{\omega}(t)|^2 dt < \infty \right\}.$$

3.1. **Heuristics.** We begin with a heuristic discussion. We now define (informally) the **partition function** by

$$Z_T^V(x,y) := \int_{H(x,y,T)} \exp\left(-\int_0^T \left(\frac{1}{2} |\dot{\omega}(t)|^2 + V(\omega(t)) \right) dt \right) \mathcal{D}\omega,$$

where we are thinking of $\mathcal{D}\omega := \prod_{t \in (0,T)} dm(\omega(t))$ where m is Lebesgue measure on $\mathbb{R}^d$. We will also let

$$H(x,T) := \left\{ \omega : [0,T] \to \mathbb{R}^d : \omega(0) = x, \quad \text{and} \quad \int_0^T |\dot{\omega}(t)|^2 dt < \infty \right\}$$

and

$$Z_T^V(x,y) := \int_{\mathbb{R}^d} Z_T^V(x,y)\, dy = \int_{H(x,T)} \exp\left(-\int_0^T \left(\frac{1}{2} |\dot{\omega}(t)|^2 + V(\omega(t)) \right) dt \right) \mathcal{D}\omega$$

where now $\mathcal{D}\omega := \prod_{t \in (0,T]} dm(\omega(t))$. If $S > 0$, then (informally)

$$\begin{aligned} Z_{S+T}^V(x,y) &= \int_{H(x,y,S+T)} \exp\left(-\int_0^{S+T} \left(\frac{1}{2} |\dot{\omega}(t)|^2 + V(\omega(t)) \right) dt \right) \mathcal{D}\omega \\ &= \int_{H(x,y,S+T)} e^{-\int_0^S \left(\frac{1}{2}|\dot{\omega}(t)|^2 + V(\omega(t))\right) dt} e^{-\int_S^{S+T} \left(\frac{1}{2}|\dot{\omega}(t)|^2 + V(\omega(t))\right) dt} \mathcal{D}\omega \\ &= \int_{\mathbb{R}^d} Z_S^V(x,z)\, Z_T^V(z,y)\, dz. \end{aligned}$$

and

$$Z_{S+T}^V(x) = \int_{\mathbb{R}^d} Z_S^V(x,z)\, Z_T^V(z)\, dz. \tag{3.1}$$

In the special case where $V = 0$, $Z_T^0(z)$ is independent of z and hence we will simply write Z_T^0 for $Z_T^0(z)$. It then follows from Eq. (3.1) that

$$Z_{S+T}^0 = Z_S^0 Z_T^0. \tag{3.2}$$

For $x \in \mathbb{R}^d$ and $T > 0$ let us define Wiener "measure," on $H(x,T)$ by

$$d\mu_{x,T}^0(\omega) := \frac{1}{Z_T^0} \exp\left(-\int_0^T \frac{1}{2} |\dot{\omega}(t)|^2 \right) \mathcal{D}\omega.$$

Lemma 3.1 (Heuristic)**.** *If $0 < t < T$ and $f : H(x,t) \to \mathbb{R}$, then*

$$\int_{H(x,T)} f\left(\omega|_{[0,t]}\right) d\mu_{x,T}^0(\omega) = \int_{H(x,t)} f(\omega)\, d\mu_{x,t}^0(\omega).$$

More generally if $g(\omega) = G\left(\omega|_{[0,T-t]}\right)$ is another function on paths, then

$$\begin{aligned} &\int_{H(x,T)} f\left(\omega|_{[0,t]}\right) g(\omega(t+\cdot) - \omega(t))\, d\mu_{x,T}^0(\omega) \\ &= \int_{H(x,t)} f(\omega)\, d\mu_{x,t}^0(\omega) \cdot \int_{H(0,T-t)} g(\omega(t+\cdot) - \omega(t))\, d\mu_{0,T}^0(\omega). \end{aligned}$$

Proof: Working informally and making use of Eq. (3.2) we have,

$$\begin{aligned} &\int_{H(x,T)} f\left(\omega|_{[0,t]}\right) d\mu_{x,T}^0(\omega) \\ &= \frac{1}{Z_T^0} \int_{H(x,T)} f\left(\omega|_{[0,t]}\right) \exp\left(-\int_0^t \frac{1}{2} |\dot{\omega}(t)|^2 \right) \exp\left(-\int_t^T \frac{1}{2} |\dot{\omega}(t)|^2 \right) \mathcal{D}\omega \\ &= \frac{1}{Z_T^0} \int_{H(x,t)} f\left(\omega|_{[0,t]}\right) \exp\left(-\int_0^t \frac{1}{2} |\dot{\omega}(t)|^2 \right) Z_{T-t}^0 \mathcal{D}\omega \\ &= \frac{1}{Z_t^0} \int_{H(x,t)} f\left(\omega|_{[0,t]}\right) \exp\left(-\int_0^t \frac{1}{2} |\dot{\omega}(t)|^2 \right) \mathcal{D}\omega \\ &= \int_{H(x,t)} f(\omega)\, d\mu_{x,t}^0(\omega). \end{aligned}$$

The "proof" of the second equation is left to the reader. Q.E.D.

Definition 3.2 (Heuristic)**.** For $t > 0$, let

$$\begin{aligned} \left(R_t^V f\right)(x) &= \frac{1}{Z_t^0} \int_{\mathbb{R}^d} Z_t^V(x,y)\, f(y)\, dy \\ &= \int_{H(x,t)} f(\omega(t))\, e^{-\int_0^t V(\omega(s)) ds} d\mu_{x,t}^0(\omega) \\ &= \int_{H(x,T)} f(\omega(t))\, e^{-\int_0^t V(\omega(s)) ds} d\mu_{x,T}^0(\omega), \end{aligned}$$

where the last expression is valid for any $T > t$.

Theorem 3.3 (Heuristic)**.** *$R_t^V : L^2\left(\mathbb{R}^d, m\right) \to L^2\left(\mathbb{R}^d, m\right)$ is a semi-group with infinitesimal generator, $\hat{H} := -\frac{1}{2}\Delta + M_V$.*

Proof: We have

$$\begin{aligned}\left(R_{t+s}^V f\right)(x) &= \frac{1}{Z_{t+s}^0}\int_{\mathbb{R}^d} Z_{t+s}^V(x,y) f(y)\,dy \\ &= \frac{1}{Z_s^0 Z_t^0}\int_{\mathbb{R}^d\times\mathbb{R}^d} Z_t^V(x,z) Z_s^V(z,y) f(y)\,dydz \\ &= \frac{1}{Z_t^0}\int_{\mathbb{R}^d} Z_t^V(x,z)\left(R_s^V f\right)(z)\,dz = R_t^V R_s^V f(x).\end{aligned}$$

Also,

$$\begin{aligned}\lim_{t\downarrow 0} R_t^V f(x) &= \lim_{t\downarrow 0}\int_{H(x,T)} f(\omega(t))\, e^{-\int_0^t V(\omega(s))ds} d\mu_{x,T}^0(\omega) \\ &= \int_{H(x,T)} f(\omega(0))\, e^{-\int_0^0 V(\omega(s))ds} d\mu_{x,T}^0(\omega) = f(x).\end{aligned}$$

Now to compute $\frac{d}{dt}R_t^V f(x)$, we have

$$\begin{aligned}\frac{d}{dt}\left(R_t^V f\right)(x) &= \frac{d}{ds}R_t^V f(x)|_{s=t} \\ &= \frac{d}{ds}|_{s=t}\int_{H(x,T)} f(\omega(s))\, e^{-\int_0^t V(\omega(s))ds} d\mu_{x,T}^0(\omega) \\ &+ \frac{d}{ds}|_{s=t}\int_{H(x,T)} f(\omega(t))\, e^{-\int_0^s V(\omega(s))ds} d\mu_{x,T}^0(\omega).\end{aligned}$$

The second term gives,

$$\begin{aligned}\frac{d}{ds}|_{s=t}&\int_{H(x,T)} f(\omega(t))\, e^{-\int_0^s V(\omega(s))ds} d\mu_{x,T}^0(\omega) \\ &= -\int_{H(x,T)} f(\omega(t))\, V(\omega(t))\, e^{-\int_0^t V(\omega(s))ds} d\mu_{x,T}^0(\omega) \\ &= -R_t^V (Vf)(x). \qquad (3.3)\end{aligned}$$

For the first term, we will suppose that $s = t+h$ with $h > 0$ and write $\Delta\omega := \omega(t+h) - \omega(t)$ and

$$f(\omega(t+h)) = f(\omega(t)) + f'(\omega(t))\,\Delta\omega + \frac{1}{2}f''(\omega(t))\,\Delta\omega^2 + O\left(\Delta\omega^3\right).$$

Then using Lemma 3.1 we find,

$$\begin{aligned}&\int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} g(\omega(t))\, u(\Delta\omega)\, d\mu_{x,T}^0(\omega) \\ &= \int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} g(\omega(t))\, u(\Delta\omega)\, d\mu_{x,T}^0(\omega) \\ &\int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} g(\omega(t))\, d\mu_{0,T}^0(\omega)\cdot\int_{H(0,T)} u(\omega(h))\, d\mu_{0,T}^0(\omega).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} f(\omega(t+h))\, d\mu_{x,T}^0(\omega) \\ &= \sum_{n=0}^\infty \int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} f^{(n)}(\omega(t))\left(\Delta\omega^n\right) d\mu_{x,T}^0(\omega) \\ &= \sum_{n=0}^\infty \int_{H(x,T)} e^{-\int_0^t V(\omega(s))ds} f^{(n)}(\omega(t))\, d\mu_{x,T}^0(\omega)\cdot\int_{H(0,T)}\omega(h)^n\, d\mu_{0,T}^0(\omega) \\ &= \sum_{n=0}^\infty \left(R_t^V f^{(n)}\right)(x)\cdot\int_{H(0,T)}\omega(h)^n\, d\mu_{0,T}^0(\omega).\end{aligned}$$

Informal Fact: If we let $c_n(h) := \int_{H(0,T)}\omega(h)^n\, d\mu_{0,T}^0(\omega)$, then $c_0(h) = 1$, $c_{\mathrm{odd}}(h) = 0$, $c_2(h) = h\sum_j e_j\otimes e_j$ and

$$\int_{H(0,T)} |\omega(h)|^n\, d\mu_{0,T}^0(\omega) = C_n\,|h|^{n/2}.$$

Therefore,

$$\begin{aligned}\int_{H(x,T)} & e^{-\int_0^t V(\omega(s))ds} f(\omega(t+h))\, d\mu_{x,T}^0(\omega) \\ &= \left(R_t^V f\right)(x) + \frac{h}{2}\left(R_t^V f^{(2)}\right)(x)\sum_j e_j\otimes e_j + O\left(h^{3/2}\right) \\ &= \left(R_t^V f\right)(x) + \frac{h}{2}\left(R_t^V \Delta f\right)(x) + O\left(h^{3/2}\right).\end{aligned}$$

Differentiating this equation in h shows,

$$\frac{d}{ds}|_{s=t}\int_{H(x,T)} f(\omega(s))\, e^{-\int_0^t V(\omega(s))ds} d\mu_{x,T}^0(\omega) = \frac{1}{2}\left(R_t^V\Delta f\right)(x). \qquad (3.4)$$

Combining Eqs. (3.3) and (3.4) shows

$$\frac{d}{dt}\left(R_t^V f\right)(x) = -R_t^V (Vf)(x) + \frac{1}{2}R_t^V(\Delta f)(x) = R_t^V\left(\hat{H}f\right)(x).$$

Q.E.D.

3.2. Rigorous Interpretation of the above Heuristics.

Remark 3.4. If $\mu_{x,T}^0$ were concentrated on differentiable paths, we would have $|\omega(t+h) - \omega(t)| \le C(\omega)\,h$ and therefore would expect that

$$\int |\omega(t+h) - \omega(t)|^2\, d\mu_{x,T}^0 \le h^2\int C^2(\omega)\, d\mu_{x,T}^0 h.$$

If this were the case the above computation would have lead to $\frac{d}{dt}\left(R_t^V f\right)(x) = -R_t^V (Vf)(x)$ and in particular when $V=0$ that $\frac{d}{dt}\left(R_t^0 f\right)(x)=0$. Thus we would have $R_t^0 f = f$ which would be absurd.

The rigorous interpretation of the following considerations is the following Feynman - Kac formula.

Theorem 3.5 (Feynman - Kac Formula). *Let $V:\mathbb{R}^d \to \mathbb{R}$ be a reasonable potential bounded from below. Then, $\hat{H} := -\frac{1}{2}\Delta + V$ is essentially self-adjoint and*

$$\left(e^{-t\hat{H}} f\right)(x) = \int_W f(x+\omega(t)) \exp\left(-\int_0^t V(\omega(s))\,ds\right) d\mu(\omega)$$

where μ is Wiener measure on

$$W := \left\{\omega \in C\left([0,\infty) \to \mathbb{R}^d\right) : \omega(0) = 0\right\}.$$

4. Harmonic Oscillator Examples

4.1. **Finite Dimensional Case.** Consider the following Newton's equations for a harmonic oscillator;

$$\ddot{q}(t) + Aq(t) = 0$$

where $A>0$ is a $N\times N$ matrix. The quantum Hamiltonian and its ground state and ground state energy, λ_0, are given by:

$$\hat{H} := -\frac{1}{2}\Delta + \frac{1}{2}(Aq,q) \text{ acting on } L^2\left(\mathbb{R}^N\right), \text{ and}$$

$$\Omega_0(q) := \sqrt[4]{\det\left(\frac{\pi}{\omega}\right)} \exp\left(-\frac{1}{2}(\omega q, q)\right)$$

$$\text{with } \hat{H}\Omega_0 = \lambda_0\Omega_0 \text{ where } \lambda_0 = \frac{1}{2}\operatorname{tr}(\omega).$$

where $\omega := \sqrt{A}$. We now pass to the ground state representation by introducing the Hilbert space

$$K = K_\omega := L^2\left(\mathbb{R}^N, \Omega_0^2(q)\,dq\right) = L^2\left(\mathbb{R}^N, \sqrt{\det\left(\frac{\pi}{\omega}\right)}\exp\left(-(\omega q,q)\right)dq\right)$$

which is unitarily equivalent to $L^2\left(\mathbb{R}^N\right)$ via, $U : L^2\left(\mathbb{R}^N\right) \to K$ defined by $Uf := \Omega_0^{-1} f$. Let

$$\begin{aligned} H_0 = U\left(\hat{H} - \lambda_0\right)U^{-1} &= -\frac{1}{2}\Delta + \nabla \ln \Omega_0 \cdot \Omega_0 \\ &= -\frac{1}{2}\Delta + \omega q\cdot\nabla. \end{aligned}$$

This last expression is the quantization of the function of the form $p^2/2 \pm i\omega q\cdot p$. Under the action of the free Newtonian dynamics, $q \to q + tp$ and $p\to p$ and therefore the term $\omega q\cdot p \to \omega(q+tp)\cdot p$. Hence we may expect to find a similarity transformation which takes, H_0 to $\omega q\cdot\nabla$ which an Euler type operator and hence easy to understand. We now carry this out in detail. (In fact what we are about to do was already done in Eq. (1.21) above and we could just use the results derived there.)

Let $L = B_{ij}\partial_i\partial_j$ with B as symmetric matrix to be chosen explicitly shortly. Observe that L is the quantum mechanical Hamiltonian for a free particle. Then

$$\frac{d}{dt}e^{tL/2}(\omega q\cdot\nabla)e^{-tL/2} = \frac{1}{2}e^{tL/2}[L,\omega q\cdot\nabla]e^{-tL/2}$$

where

$$\frac{1}{2}[L,\omega q\cdot\nabla] = B_{ij}\omega e_j\cdot\nabla\partial_i = \omega B e_i \cdot\nabla\partial_i = \partial_{\omega B e_i}\partial_{e_i}.$$

Therefore,

$$e^{tL/2}(\omega q\cdot\nabla)e^{-tL/2} = \omega q\cdot\nabla + t\partial_{\omega B e_i}\partial_{e_i}$$

and hence it we choose $B = \omega^{-1}$, then

$$e^{tL/2}(\omega q\cdot\nabla)e^{-tL/2} = \omega q\cdot\nabla + t\partial_{e_i}\partial_{e_i} = \omega q\cdot\nabla + t\Delta$$

and therefore,

$$e^{tL/2}(H_0)e^{-tL/2} = -\frac{1}{2}\Delta + \omega q\cdot\nabla + t\Delta.$$

Taking $t=2$ then implies

$$e^L H_0 e^{-L} = \omega q\cdot\nabla$$

or equivalently put

$$H_0 = e^{-L}\omega q\cdot\nabla e^L$$

where $L := \omega_{ij}^{-1}\partial_i\partial_j$. Let $\{\xi_j\}_{j=1}^N$ be an orthonormal basis of Eigenvectors of ω (or equivalently A) with $\omega\xi_j = \omega_j\xi_j$. Now suppose that $h_\alpha(x) = e^{-L}\xi^\alpha$ where we are writing ξ^α to be the polynomial function on $\mathbb{R}^N$ defined by $\xi^\alpha(q) := \prod_{j=1}^N (\xi_j, q)^{\alpha_j}$. Then

$$\omega q\cdot\nabla = (q,\xi_j)\,\omega\xi_j\cdot\nabla = (q,\xi_j)\,\omega_j\xi_j\cdot\nabla = \omega_j(q,\xi_j)\,\partial_{\xi_j}$$

so that

$$\begin{aligned} \omega q\cdot\nabla\xi^\alpha &= \omega_j(q,\xi_j)\,\partial_{\xi_j}\xi^\alpha = \omega_j(q,\xi_j)\,\alpha_j\xi^{\alpha-e_j}(q) = \omega_j\alpha_j\xi^\alpha(q) \\ &= (\omega\cdot\alpha)\,\xi^\alpha. \end{aligned}$$

Hence it follows that

$$H_0 h_\alpha = e^{-L}\omega q\cdot\nabla e^L h_\alpha = e^{-L}\omega q\cdot\nabla\xi^\alpha = e^{-L}(\omega\cdot\alpha)\,\xi^\alpha = (\omega\cdot\alpha)\,h_\alpha.$$

Thus we have now found a total orthonormal subset of eigenvectors for H_0, $\{h_\alpha\}_{\alpha\in\mathbb{N}_0}$ with corresponding eigenvalues, $\sigma\left(\hat{H}_0\right) = \{\omega\cdot\alpha\}_{\alpha\in\mathbb{N}_0}$. The spectrum of $\hat{H}$ is thus given by

$$\sigma\left(\hat{H}\right) = \left\{\omega\cdot\alpha + \frac{1}{2}\operatorname{tr}(\omega)\right\}_{\alpha\in\mathbb{N}_0}.$$

4.2. **An Infinite Dimensional Example.** Let us now go back to the path integral expressions like that in Eq. (1.18). Here we will consider the related "measure,"

$$d\mu(\varphi) = \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{[0,T]\times S^1}\left(|\nabla\varphi|^2 + m^2\varphi^2\right)dx\right)\mathcal{D}\varphi.$$

By viewing $\varphi(t,\theta)$ as a function of t with values in $L^2(S^1)$, the above expression may be written as

$$d\mu(\varphi) = \frac{1}{Z}\exp\left(-\int_{[0,T]}\left(\frac{\|\dot\varphi(t)\|^2}{2} + V(\varphi(t))\right)dt\right)\mathcal{D}\varphi \tag{4.1}$$

where $\|\varphi\|^2 := \int_{S^1}\varphi^2(\theta)\,d\theta$ and

$$V(f) := \|\partial_\theta f\|^2 + m^2\|f\|^2 = \left(\left(-\partial_\theta^2 + m^2\right)f, f\right).$$

Thus it is reasonable to associate μ to the the elliptic differential operator,

$$\hat{H} := -\frac{1}{2}\Delta_{L^2(S^1)} + M_V \tag{4.2}$$

which is the Hamiltonian associated to an infinite dimensional quantum mechanical oscillator. Using the considerations developed above, we have

$$A = \left(-\partial_\theta^2 + m^2\right), \text{ and } \lambda_0 = \frac{1}{2}\operatorname{tr}(A) = \infty.$$

Hence we conclude, formally, that $\hat{H} \geq \infty I$ which is of course nonsense. We must renormalize $\hat{H}$ to take care off this by subtracting of this ground state energy. (The idea here is that the potential energy is only defined up to an additive constant and hence subtracting λ_0 (albeit ∞) from $\hat{H}$ does not change the physics being described.) It is also to our benefit to go to the ground state representation where we take the quantum mechanical Hilbert space to be, informally,

$$L^2\left(H_{1/2}(S), d\nu_0(f) := \text{“}\frac{1}{Z}\exp\left(\sqrt{(-\partial_\theta^2 + m^2)}f, f\right)_{L^2(S)}\mathcal{D}f\text{”}\right).$$

Again this has to be made sense of using the considerations introduced above. There is of course much more to be said, but it will not be done here and now.

5. Path Integral Quantization

Suppose that $V : \mathbb{R}^N \to \mathbb{R}$ is a smooth potential such that $V(x) \to \infty$ as $|x| \to \infty$. Let $H_0 := -\frac{1}{2}\Delta + V$ and λ_0 denoted the lowest eigenvalue of H_0 and Ω_0 be the corresponding eigenfunction which we may assume is strictly positive. The renormalized Hamiltonian is now defined to be $H = H_0 - \lambda_0 I$, so that $H \geq 0$, $0 = \inf\sigma(H)$ and $H\Omega_0 = 0$ with $\Omega_0 > 0$.

The goal now is to make sense out of the following expression

$$d\mu(\omega) = \frac{1}{Z}e^{-\int_{-\infty}^{\infty}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\mathcal{D}\omega, \tag{5.1}$$

where $\mathcal{D}\omega$ is "Lebesgue measure", Z is a normalization constant chosen so that μ becomes probability measure. Notice that $\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))$ is the classical energy of a system with mass $m = 1$ in subject to a the force $-\nabla V$.

To make sense of Eq. (5.1), let us begin by truncating the time interval. To this end let $T > 0$ and α and β be two probability densities (or more generally measures) on $\mathbb{R}^N$. Let begin by considering the informal expression

$$d\mu_T(\omega) = \frac{1}{Z_T}e^{-\int_{-T}^{T}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\alpha(\omega(-T))\beta(\omega(T))\mathcal{D}\omega, \tag{5.2}$$

where now

$$Z_T = \int_{C([-T,T]\to\mathbb{R}^N)} e^{-\int_{-T}^{T}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\alpha(\omega(-T))\beta(\omega(T))\mathcal{D}\omega.$$

The first observation we should make is that (at least formally) in Eq. (5.1) and (5.2) may replace V by $V - \lambda_0$ without changing the measure μ_T and μ. So we now assume this has been done and we will rename $V - \lambda_0$ by V. Let $W^T := C([-T,T] \to \mathbb{R}^N)$, $\mathcal{P} = \{-T = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[-T,T]$ and

$$W_{\mathcal{P}}^T = \{\omega \in W^T : \omega''(t) = 0 \text{ for all } t \notin \mathcal{P}\}.$$

Given a function $F : W^T \to \mathbb{R}$, let

(5.3)

$$\mu_T^{\mathcal{P}}(F) := \frac{1}{Z_T^{\mathcal{P}}}\int_{W_{\mathcal{P}}^T} e^{-\int_{-T}^{T}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}F(\omega)\alpha(\omega(-T))\beta(\omega(T))d\lambda_{\mathcal{P}}(\omega)$$

where as usual $Z_T^{\mathcal{P}}$ is a normalization constant so as to make $\mu_T^{\mathcal{P}}$ a probability measure. Let ν_T denote Wiener measure on W^T with initial distribution α and end point distribution given by β. For $\omega \in W^T$, let $\omega_{\mathcal{P}}$ denote the unique path in W_P^T such that $\omega_{\mathcal{P}} = \omega$ on $\mathcal{P}$. Then we may write $\mu_T^{\mathcal{P}}$ as

$$\mu_T^{\mathcal{P}}(F) := \frac{\int_{W^T} e^{-\int_{-T}^{T} V(\omega_{\mathcal{P}}(t))dt}F(\omega_{\mathcal{P}})d\nu_T(\omega)}{\int_{W^T} e^{-\int_{-T}^{T} V(\omega_{\mathcal{P}}(t))dt}d\nu_T(\omega)}. \tag{5.4}$$

To get a rigorous interpretation of Eq. (5.2), let us pass to the limit $|\mathcal{P}| \to 0$ in Eq. (5.4) to find for continuous and bounded F that

$$\mu_T(F) = \lim_{|\mathcal{P}|\to 0}\mu_T^{\mathcal{P}}(F) = \frac{\int_{W^T} e^{-\int_{-T}^{T} V(\omega(t))dt}F(\omega)d\nu_T(\omega)}{\int_{W^T} e^{-\int_{-T}^{T} V(\omega(t))dt}d\nu_T(\omega)}. \tag{5.5}$$

Suppose that $F(\omega) = f(\omega(s_1), \omega(s_2), \ldots, \omega(s_k))$ where $-T < s_1 < s_2 < \cdots < s_k < T$. Making use of the Feynman Kac formula, we have that

$$\mu_T(F) =$$

$$= \frac{\int_{(\mathbb{R}^N)^{k+2}} f(x_1, x_2, \ldots, x_k)\alpha(x_0)\beta(x_{k+1}) \prod_{i=1}^{k+1} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_0 dx_1 \ldots dx_{k+1}}{\int_{(\mathbb{R}^N)^{k+2}} \alpha(x_0)\beta(x_{k+1}) \prod_{i=1}^{k+1} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_0 dx_1 \ldots dx_{k+1}}$$
$$= \frac{\int_{(\mathbb{R}^N)^{k+2}} f(x_1, x_2, \ldots, x_k)\alpha(x_0)\beta(x_{k+1}) \prod_{i=1}^{k+1} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_0 dx_1 \ldots dx_{k+1}}{(\alpha, e^{-2TH}\beta)},$$

where $s_0 = -T$, $s_{k+1} = T$, $\Delta_i s := s_i - s_{i-1}$, and $p^H_t(x, y)$ is the integral kernel of e^{-tH}. Making use of the fact that $e^{-\tau H}\alpha \to (\alpha, \Omega_0)\Omega_0$ as $\tau \to \infty$, we find using the previous equation that

$$\begin{aligned}\mu(F) &= \lim_{T\to\infty} \mu_T(F)\\ &= \frac{\left[\begin{array}{c}(\alpha, \Omega_0)(\beta, \Omega_0) \int_{(\mathbb{R}^N)^k} f(x_1, x_2, \ldots, x_k)\\ \times\Omega_0(x_1)\Omega_0(x_k) \prod_{i=2}^{k} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_1 dx_2 \ldots dx_k\end{array}\right]}{(\alpha, \Omega_0)(\beta, \Omega_0)}\\ &= \int_{(\mathbb{R}^N)^k} f(x_1, x_2, \ldots, x_k)\Omega_0(x_1)\Omega_0(x_k) \prod_{i=2}^{k} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_1 dx_2 \ldots dx_k.\end{aligned}$$

The conclusion of the above computations may be summarized as follows.

Theorem 5.1. *The informal expression in Eq. (5.1) should satisfy*

$$\begin{aligned}&\frac{1}{Z}\int_W F(\omega) e^{-\int_{-\infty}^{\infty}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\mathcal{D}\omega\\ &\quad= \int_{(\mathbb{R}^N)^{k+2}} f(x_1, x_2, \ldots, x_k)\Omega_0(x_1)\Omega_0(x_k) \prod_{i=2}^{k} p^H_{\Delta_i s}(x_{i-1}, x_i)\, dx_1 dx_2 \ldots dx_k\end{aligned}$$

where $F(\omega) = f(\omega(s_1), \omega(s_2), \ldots, \omega(s_k))$ and $s_1 < s_2 < \cdots < s_k$. In particular we have that

$$\frac{1}{Z}\int_W f(\omega(0)) e^{-\int_{-\infty}^{\infty}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\mathcal{D}\omega = \int_{\mathbb{R}^N} f(x)\Omega_0^2(x)dx$$

so that the time 0 distribution of μ has a density equal to the square of the ground state of H. Moreover,

$$\begin{aligned}&\frac{1}{Z}\int_W f(\omega(0))g(\omega(t)) e^{-\int_{-\infty}^{\infty}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\mathcal{D}\omega\\ &\qquad= \int_{\mathbb{R}^N\times\mathbb{R}^N} f(x)g(y)\Omega_0(x)\Omega_0(y)p^H_t(x, y)dxdy\\ &\qquad= (f\Omega_0, e^{-tH}(g\Omega_0)).\end{aligned}$$

Differentiating this equation at $t = 0$ gives,

$$(f\Omega_0, H(g\Omega_0)) = -\frac{d}{dt}|_0 \frac{1}{Z}\int_W f(\omega(0))g(\omega(t)) e^{-\int_{-\infty}^{\infty}\left(\frac{1}{2}|\omega'(t)|^2 + V(\omega(t))\right)dt}\mathcal{D}\omega$$

from which we recover the Hamiltonian of the system.

6. Gross' QFT Notes Introduction

(These are notes of Leonard Gross which were originally written sometime around 1975.) The purpose of these notes is to show how the quantization of a classical field leads to the study of an elliptic differential operator in infinitely many variables. We will see here, in a simple instance, how such a differential operator arises as the Hamiltonian operator for the quantized field. The following exposition is distilled from the many similar discussions in the physics literature. See e.g [vN,p.] or [D, p].

6.1. The harmonic oscillator.

Recall that Newton's equation, $F = ma$, reduces, in the case of a harmonic oscillator, to

$$-kx = m\ddot{x} \tag{6.1}$$

where x is the oscillator position, $\dot{x} = dx/dt$, and k is the spring constant. The force F is $-\text{grad}\, V$, with $V = \frac{k}{2}x^2$. The energy of the oscillator is therefore

$$E = \frac{1}{2}m\dot{x}^2 + \frac{k}{2}x^2$$

or

$$E = \frac{1}{2m}p^2 + \frac{k}{2}x^2, \tag{6.2}$$

where $p = mv = m\dot{x}$ is the momentum.

Quantization of this system yields the Hamiltonian

$$H = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{k}{2}x^2, \tag{6.3}$$

which is to be interpreted as a self–adjoint operator on $L^2((-\infty, \infty); dx)$ (and which is obtained from (6.2) by the usual substitution $p \to \frac{1}{i}\frac{\partial}{\partial x}$, $x \to$ mult. by x).

6.1.1. *n harmonic oscillators.*

Similarly the Hamiltonian for a system consisting of n independent harmonic oscillators of masses m_j and spring constants k_j may be derived from the corresponding Newton's equations,

$$m_j \frac{d^2 x_j}{dt^2} = -k_j x_j \quad j = 1, \ldots, n. \tag{6.4}$$

The Hamiltonian is given by

$$H = \sum_{j=1}^{n}\left(-\frac{1}{2m_j}\frac{\partial^2}{\partial x_j^2} + \frac{k_j}{2}x_j^2\right). \tag{6.5}$$

H is to be interpreted as a self–adjoint operator on a suitable domain in $L^2(\mathbb{R}^n; d^n x)$ (e.g., it may be defined as the closure of its restriction to $C_c^\infty(\mathbb{R}^n)$).

6.2. **A quantized field; informalities.** We will show now how the quantization of a classical (non-interacting) field may be regarded as just an infinite dimensional version of the preceding procedure. For simplicity we consider a single component u of the electromagnetic field. We will assume there are no charges or currents present. I.e., it is not interacting with anything. Then the field $u(x,t)$ satisfies the homogeneous wave equation $\Delta u = \frac{\partial^2 u}{\partial t^2}$. Whereas quantum mechanics arises from quantizing an ordinary differential equation, namely Newton's equation, quantum field theory arises from quantizing a partial differential equation (the field equation). In order to keep the exposition very elementary at this stage we will reduce the number of space dimensions to one in the preceding field equation, thereby obtaining the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{6.6}$$

which is the equation for a vibrating string. Moreover the main ideas will not be altered if we consider the string only on the interval $(0,\pi)$ and assume it is fixed at the endpoints. I.e.,

$$u(0,t) = 0 \quad u(\pi,t) = 0 \quad \text{for all} \quad t. \tag{6.7}$$

In accordance with the method of separation of variables a general solution of (6.6) and (6.7) can be written in the form

$$u(x,t) = \sum_{j=1}^{\infty} q_j(t) u_j(x) \tag{6.8}$$

where $u_j(x)$ and $q_j(t)$ each satisfy ordinary differential equations and u_j is zero at the endpoints. Specifically we may take

$$u_j(x) = \sqrt{\frac{2}{\pi}} \sin jx \quad j = 1, 2, \ldots$$

which is a normalized eigenfunction for d^2/dx^2 with Dirichlet boundary conditions and with eigenvalue $-j^2$. That is, one has $\partial^2 u_j/\partial x^2 = -j^2 u_j$. So the functions q_j satisfy

$$\frac{d^2 q_j}{dt^2} = -j^2 q_j \quad j = 1, 2, \ldots\ . \tag{6.9}$$

All of this follows from (6.8), the orthonormality of the u_j and the equations

$$\frac{\partial^2 u}{\partial x^2} = \sum_{j=1}^{\infty} -j^2 q_j(t) u_j(x),$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{\infty} \ddot{q}_j(t) u_j(x).$$

in accordance with the usual procedure of the method of separation of variables.

Of course $u(x,t)$ is determined by the two sequences $\{q_j(0)\}$ and $\{\dot{q}_j(0)\}$ which, in view of (6.8), are determined by the initial values $u(x,0)$ and $\dot{u}(x,0)$. Comparison of equations (6.9) with (6.4) shows that the boundary value problem (6.6), (6.7) is equivalent to a mechanical system consisting of infinitely many harmonic oscillators with masses m_j all equal to one and spring constants $k_j = j^2$. The canonical coordinates are $\{q_j\}_{j=1}^{\infty}$ and $\{p_j = m_j \dot{q}_j = \dot{q}_j\}_{j=1}^{\infty}$. According to the principles of quantum mechanics one should quantize this system by taking the Hilbert space to be

$$\mathcal{K} = L^2(\mathbb{R}^\infty, dq_1 dq_2 \cdots) \tag{6.10}$$

and the Hamiltonian to be (in analogy to (6.5))

$$H = \sum_{j=1}^{\infty} \frac{1}{2}\left(-\frac{\partial^2}{\partial q_j^2} + \omega_j^2 q^2\right) \tag{6.11}$$

where $\omega_j = j$.

This definition of $\mathcal{K}$ is meaningless, however, because of the appearance of infinite dimensional Lebesgue measure. Further, the expression for H contains problems of its own, as we will see.

If Q_j denotes the quantized version of q_j (informally Q_j is multiplication by q_j on $\mathcal{K}$) the principles of quantum mechanics assert, in view of (6.8), that the quantized field φ at time zero is given by

$$\varphi(x,0) = \sum_{j=1}^{\infty} Q_j u_j(x).$$

The field is therefore an operator valued function. Actually this series of operators does not converge for each x. However, as is known, the series converges in the sense of distributions to an operator valued distribution $\varphi(f,0)$ which corresponds informally to

$$\int_0^\pi \varphi(x,0) f(x) dx$$

where f is a test function.

In order to rescue the preceding discussion of the quantized field we return briefly to the harmonic oscillator.

6.3. **Ground state transformation.**

6.3.1. *One oscillator.* Consider a single harmonic oscillator with $m = 1$ and $k = \omega^2$. Its Hamiltonian is

$$H' = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\omega^2 x^2 \quad \text{on} \quad L^2((-\infty,\infty); dx). \tag{6.12}$$

Let

$$\psi(x) = (\omega/\pi)^{1/4} e^{-\omega x^2/2}.$$

It is straightforward to verify that

(1) $\int_{-\infty}^{\infty} \psi(x)^2 dx = 1$ and
(2) $H'\psi = (\omega/2)\psi$

So ψ is a normalized eigenfunction for H' and, since ψ is positive, $\omega/2$ is the lowest eigenvalue of H' and ψ is the ground state of H'. We now apply JACOBI'S GROUND STATE TRANSFORMATION. Let

$$d\mu(x) = \psi(x)^2 dx. \tag{6.13}$$

Then μ is a probability measure on $\mathbb{R}$. Define

$$U : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, \mu)$$

by

$$(Uf)(x) = f(x)/\psi(x).$$

U is clearly unitary. Moreover one can compute that

$$U(H' - \omega/2)U^{-1} = (1/2)(-d^2/dx^2 + \omega x d/dx). \tag{6.14}$$

The computation can be carried out easily by observing that $(U^{-1}g)(x) = g(x)\psi(x)$ so that

$$\begin{aligned}(H'U^{-1}g)(x) &= (1/2)(-d^2/dx^2 + \omega^2x^2)(g(x)\psi(x)) \\ &= (\omega/2)g(x)\psi(x) - (1/2)g'(x)\psi'(x) - (1/2)g''(x)\psi(x) \\ &= [(\omega/2)g + (1/2)\omega x g'(x) - (1/2)g''(x)]\psi(x),\end{aligned}$$

which proves (6.14).

Define

$$\hat{H} = (1/2)\left(-d^2/dx^2 + \omega x d/dx\right) \qquad \text{on } L^2(\mathbb{R}, \mu) \tag{6.15}$$

So

$$\hat{H} = U(H' - \omega/2)U^{-1} \tag{6.16}$$

The virtue of unitarily transforming $H' - \omega/2$ to $\hat{H}$ is that $\hat{H}$ happens to be the operator associated to the Dirichlet form for μ. To be somewhat precise about this note that an integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}} f'(x)\overline{g}'(x)d\mu(x) &= \int_{\mathbb{R}} f'(x)\overline{g}'(x)\psi(x)^2 dx \\ &= \int_{\mathbb{R}} (-f''(x) + \omega x f'(x))\overline{g}(x)\psi(x)^2 dx \\ &= 2(\hat{H}f, g)_{L^2(\mu)}.\end{aligned}$$

So we have the identity

$$(\hat{H}f, g)_{L^2(\mu)} = (1/2)\int_{\mathbb{R}} f'(x)\overline{g}'(x)d\mu(x) \tag{6.17}$$

Clearly $(\hat{H}f, f) \geq 0$. Moreover the constant functions are in $L^2(\mu)$ and $\hat{H}1 = 0$. So inf(spectrum $\hat{H}$) $= 0$. Of course this just reflects the fact that inf(spectrum $(H' - \omega/2)) = 0$, which we already knew.

6.3.2. *n oscillators.* Let us return now to a system of n independent harmonic oscillators all of mass one and spring constants ω_j^2. Specializing (6.5) to this case we find the Hamiltonian to be

$$H_n = (1/2)\sum_{j=1}^{n}(-\partial^2/\partial x_j^2 + \omega_j^2 x_j^2) \quad \text{acting in} \quad L^2(\mathbb{R}^n, dx) \tag{6.18}$$

Let

$$\psi_j(x_j) = (\omega_j/\pi)^{1/4} e^{-\omega_j^2 x_j^2/2}$$

and

$$\psi(x) = \Pi_{j=1}^n \psi_j(x_j), \quad x = (x_1, \ldots, x_n)$$

Then clearly

$$H_n\psi = \left[\frac{1}{2}\sum_{j=1}^{n}\omega_j\right]\psi \tag{6.19}$$

because H_n is a sum of independent harmonic oscillator Hamiltonians. So ψ is the ground state of H_n and the "zero point energy" (i.e. inf spectrum H) is

$$E_n \equiv (1/2)\sum_{j=1}^{n}\omega_j. \tag{6.20}$$

Define

$$d\mu_n(x) = \Pi_{j=1}^n[\psi_j(x)^2 dx_j] \quad \text{on} \quad \mathbb{R}^n \tag{6.21}$$

Then μ_n is a probability measure on $\mathbb{R}^n$ and the map $f \to Uf = f/\psi$ is again unitary from $L^2(\mathbb{R}^n, dx)$ onto $L^2(\mathbb{R}^n, \mu_n)$. Just as for a single oscillator one sees that

$$U(H_n - E_n)U^{-1} = (1/2)\sum_{j=1}^{n}(-\partial^2/\partial x_j^2 + \omega_j x_j \partial/\partial x_j). \tag{6.22}$$

So define

$$\hat{H}_n = (1/2)\sum_{j=1}^{n}(-\partial^2/\partial x_j^2 + \omega_j x_j \partial/\partial x_j) \quad \text{acting in} \quad L^2(\mathbb{R}^n, \mu_n). \tag{6.23}$$

Then

$$UH_nU^{-1} = \hat{H}_n + E_n. \tag{6.24}$$

$\hat{H}_n$ differs from the sum of the first n terms in (6.11) in two ways. First, it incorporates the subtraction of the zero point energy E_n, which does not affect the physics. And second, it incorporates the unitary transform U, which also does not affect the physics.

6.4. **Back to the quantized fields.** We can see now that the operator H "defined" in (6.10) is bounded below by $(1/2)\sum_{j=1}^{\infty}\omega_j$, which diverges because $\omega_j = j$ in that field theory. So we can say, informally , that $H \geq +\infty$, which shows that H is at best meaningless. The customary resolution of this problem goes like this. Any potential is defined only up to an additive constant because only grad V has direct physical meaning. Thus there is no change in the physics if we subtract the infinite constant $(1/2)\sum_{j=1}^{\infty}\omega_j$ from the operator H.

We may therefore attempt to give the infinite sum in (6.11) a mathematical interpretation by passing to the limit $n \to \infty$ in the equivalent system (6.23)

To this end note first that the measure μ_n in (6.21) is just a product of probability measures, an infinite product of which is a perfectly respectable measure. Define

$$d\mu(q) = \Pi_{j=1}^{\infty}[\psi_j(q_j)^2 dq_j] \quad \text{on} \quad \mathbb{R}^{\infty} \tag{6.25}$$

Let

$$\hat{H} = (1/2)\sum_{j=1}^{\infty}(-\partial^2/\partial q_j^2 + \omega_j x_j \partial/\partial q_j) \tag{6.26}$$

Since there are no zeroth order terms in $\hat{H}$ (unlike (6.18)) the infinite sum makes perfectly good sense when applied to many functions on the infinite product space. For example if $f : \mathbb{R}^{\infty} \to \mathbb{C}$ depends on only finitely many coordinates, say $f(q) = g(q_1, \ldots, q_n)$, with $g \in C_c^{\infty}(\mathbb{R}^{\infty})$, then there are only finitely many nonzero terms in the sum for $\hat{H}f$. Such functions are dense in $L^2(\mathbb{R}^n, \mu)$. So $\hat{H}$ is densely defined by the formula (6.26). Thus by subtracting the infinite zero point energy from (6.11) and unitarily transforming to the ground state measure μ we have given a meaning to (6.11) as a densely defined operator in $L^2(\mathbb{R}^n, \mu)$.

The operator $\hat{H}$ is particularly nicely related to the measure μ. Just as in (6.17) the operator $\hat{H}_n$ in (6.23) clearly satisfies

$$(\hat{H}_n f, g)_{L^2(\mu_n)} = (1/2)\int_{\mathbb{R}^{\infty}}(\nabla f \cdot \nabla \overline{g})d\mu \tag{6.27}$$

Similarly the field theoretic Hamiltonian satisfies

$$(\hat{H}f, g)_{L^2(\mathbb{R}^n,\mu)} = (1/2)\int_{\mathbb{R}^{\infty}}\sum_{j=1}^{\infty}(\partial f/\partial q_j)(\partial \overline{g}/\partial q_j)d\mu(x) \tag{6.28}$$

If one wished to pursue the functional analytic questions left untouched in this discussion, such as whether $\hat{H}$ actually has a self-adjoint version in $L^2(\mu)$, the description of $\hat{H}$ as the operator associated to a Dirichlet form offers a quick and easy mechanism to do this because Dirichlet forms in finite and infinite dimensions are well understood.

Let us try to understand better, but at an informal level, the infinite product measure μ defined in (6.25). Write $Dq = \Pi_{j=1}^{\infty}dq_j$ for Lebesgue "measure" on $\mathbb{R}^{\infty}$. Then (6.25) gives

$$d\mu(x) = [\Pi_{j=1}^{\infty}(\omega_j/\pi)^{1/2}][e^{-\sum_{j=1}^{\infty}\omega_j q_j^2}]Dq. \tag{6.29}$$

In the case of primary interest to us we have $\omega_j = j$. So the first of the three factors in (6.29) is infinite. The second factor happens to be zero a.e. with respect to μ. And the third factor is meaningless (though there have been some attempts to give a useful interpretation to Dq.) It is therefore particularly edifying that the product of all three factors makes perfectly good sense as a measure on $\mathbb{R}^{\infty}$

At time $t = 0$ the equation (6.8) reads

$$v(x) \equiv u(x, 0) = \sum_{j=1}^{\infty} q_j u_j(x). \tag{6.30}$$

Now denote by B the nonnegative self-adjoint operator on $L^2((0, 2\pi))$ which is the square root of the Dirichlet Laplacian. Thus $B^2 = -d^2/dx^2$ with Dirichlet boundary conditions. Then

$$Bu_j = ju_j. \tag{6.31}$$

Hence

$$(Bv, v)_{L^2((0,2\pi))} = \sum_{j=1}^{\infty} jq_j^2. \tag{6.32}$$

Moreover, since the q_j are Cartesian coordinates in the Hilbert space $\operatorname{Re} L^2((0, 2\pi))$, (6.29) may be written in terms of the initial data v instead of the $q_j s$. So, writing Dv for Lebesgue "measure" instead of Dq, we may write

$$d\mu(v) = Z^{-1}e^{-(Bv,v)_{L^2((0,2\pi))}}Dv, \tag{6.33}$$

where Z^{-1} is the normalization constant (which is actually $+\infty$.) Let us transform this informal notation for the well defined measure μ once more, taking into account the fact that $(Bv, v) = \|B^{1/2}v\|^2$, which is exactly the Sobolev norm for the Sobolev space $H_{1/2}((0, 2\pi))$ (with Dirichlet boundary conditions.) Thus we may write

$$d\mu(v) = Z^{-1}e^{-\|v\|^2_{H_{1/2}}}Dv \tag{6.34}$$

The informal extension of the preceding discussion to higher dimensions, with or without a box, is straight forward. The important thing to keep in mind is that the quadratic form in the exponent in (6.29) comes from the square root of the operator whose eigenvalues were ω_j^2. For example if we begin with the Klein-Gordon equation

$$\partial^2 u/\partial t^2 = -(m^2 - \Delta)u \quad \text{on} \quad \mathbb{R}^3 \times \mathbb{R} \tag{6.35}$$

as our classical field equation then the preceding informal discussion suggests that the ground state measure for the quantized field should be (informally)

$$d\nu(v) = Z^{-1} e^{-((m^2-\Delta)^{1/2}v,v)_{L^2(\mathbb{R}^3)}} Dv \tag{6.36}$$

where Dv is Lebesgue "measure" on $\mathrm{Re}\, L^2(\mathbb{R}^3)$. Further, the preceding discussion suggests that the Hamiltonian should be determined by

$$(Hf,g)_{L^2(\nu)} = (1/2) \int_{\mathrm{Re}\, L^2(\mathbb{R}^3)} (\nabla f(v), \nabla \bar{g}(v))_{L^2(\mathbb{R}^3)} d\nu(v). \tag{6.37}$$

Gaussian measures such as (6.36) are well understood. One knows that in actuality the measure ν lives as a countably additive probability measure on "large" sets such as $\mathrm{Re}\, \mathcal{S}'(\mathbb{R}^3)$, but not on "small" sets such as $\mathrm{Re}\, L^2(\mathbb{R}^3)$. In any case let us note that the determining norm for the Gauss measure is the Sobolev $H_{1/2}$ norm since

$$((m^2 - \Delta)^{1/2} v, v)_{L^2(\mathbb{R}^3)} = \|(m^2 - \Delta)^{1/4} v\|^2_{L^2(\mathbb{R}^3)} = \|u\|^2_{H_{1/2}}.$$

The appearance of the $H_{1/2}$ norm in this heuristic discussion is consistent with the (rigorous) appearance of its dual norm $H_{-1/2}$ in Euclidean quantum field theory for the space of test functions [Book by Glimm and Jaffe], [Book Euclidean QFT, by Barry Simon]. See also e.g. L. Gross's paper in "Functional integration and its applications" Ed. A.M. Arthurs (1974) Oxford U. Press (Cumberland Lodge Conference) for further discussion of the connection at a rigorous level.

6.4.1. *Comparison with physics literature.* In comparing the previous discussion with the standard treatments in physics texts one should be aware of a change of coordinates that is very often made that may obscure the comparison. Let

$$d\gamma(s) = \frac{1}{\sqrt{\pi}} e^{-s^2} ds. \tag{6.38}$$

Then $\gamma((-\infty,\infty)) = 1$. Referring back to the measure μ in (6.13) one sees that the change of coordinates $x \to s = \omega^{1/2} x$ transforms the measure μ into γ. Moreover the Hamiltonian $\hat{H}$ of (6.15) is transformed into

$$\tilde{H} = (1/2)\omega \left\{ -\frac{d^2}{ds^2} + s\frac{d}{ds} \right\}. \tag{6.39}$$

Similarly the change of coordinates $x_j \to s_j = \omega_j^{1/2} x_j$ changes the measure in (6.21) into $\gamma^n \equiv \gamma \times \cdots \times \gamma$ (n times) and the Hamiltonian $\hat{H}_n$ in (6.23) into

$$\tilde{H}_n = (1/2) \sum_{j=1}^n \omega_j \left\{ -\frac{\partial^2}{\partial s_j^2} + s_j \frac{\partial}{\partial s_j} \right\}. \tag{6.40}$$

There is a virtue in using the form (6.38) because it has the convenient expression

$$\sum_{j=1}^n \omega_j a_j^* a_j$$

in terms of the annihilation operators a_j in $L^2(\mathbb{R}^n, \gamma^n)$. However when this procedure is used on the coordinates q_j in (6.30) the new coordinates are no longer orthonormal coordinates in $L^2((0, 2\pi))$ and the Sobolev space $H_{1/2}$ gets lost. Nevertheless this change of coordinates is convenient for other purposes and is frequently made in the name of convenience. See e.g. [Von Neumann's book, page 265], [Dirac's book p. 136], [Bjorken and Drell, Vol.2 page 8], [Jauch and Rohrlich, page 35 Equation 2-62], [Boguliubov and Shirkov, page 32]

We have developed now the Hamiltonian for a field which does not interact with charges, currents or other sources. When such sources or self–interactions are present the Hamiltonian H_0 will be modified by the presence of an additional term V which is usually a multiplication operator. Thus the theory of interacting fields is the theory of elliptic differential operators H in infinitely many variables, of which a Schrödinger operator

$$H = H_0 + V$$

in infinitely many dimensions is typical.

7. Appendix – Some Wiener Space Results

Let $W = \{\omega \in C([0,1] \to \mathbb{R}) : \omega(0) = 0\}$ and let H denote the set of functions $h \in W$ which are absolutely continuous and satisfy $(h,h) = \int_0^1 |h'(s)|^2 ds < \infty$. The space H is called the Cameron-Martin space and is a Hilbert space when equipped with the inner product

$$(h,k)_H = \int_0^1 h'(s)k'(s)ds \text{ for all } h,k \in H.$$

The space W is a Banach space when equipped with the sup-norm,

$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

Proposition 7.1. *Let $G(s,t) = \min(s,t)$. Then G is the* ***reproducing kernel*** *for H, i.e. $(G(s,\cdot),h) = h(s)$ for all $s \in [0,1]$.*

Proof: The proof follows from the fundamental theorem of calculus for absolutely continuous functions,

$$(G(s,\cdot),h)=\int_0^1 1_{t\le s}h'(t)dt=h(s).$$

Q.E.D.

Corollary 7.2. *Let $\{h_n\}_{n=1}^\infty$ be **any** Orthonormal. basis for H. Then*

$$\sum_{n=1}^\infty h_n(s)h_n(t)=G(s,t).$$

Proof: The proof is simply Bessel's equality,

$$\sum_{n=1}^\infty h_n(s)h_n(t)=\sum_{n=1}^\infty (G(s,\cdot),h_n)(G(t,\cdot),h_n)$$
$$=(G(s,\cdot),G(t,\cdot))=G(s,t).$$

Q.E.D.

Proposition 7.3 (A Sobolev Theorem)**.** *The inclusion map $i:H\to W$ is continuous and in fact*

$$\|h\|_W\le\|h\|_H \text{ for all } h\in H.$$

Proof: By Proposition 7.1, for $s\in[0,1]$,

$$|h(s)|=|(G(s,\cdot),h)_H|\le\|G(s,\cdot)\|_H\,\|h\|_H\,.$$

This proves the Proposition since

$$\|G(s,\cdot)\|_H^2=\int_0^1(1_{t\le s})^2\,dt=s\le 1.$$

Q.E.D.

8. After thoughts?

Proposition 8.1. *Let $a=(a_1,a_2,\dots)\in(0,\infty)^{\mathbb{N}}$ be a sequence such $\sum_{i=1}^\infty a_i<\infty$ and let $a^{-1}:=(1/a_1,1/a_2,\dots)\in(0,\infty)^{\mathbb{N}}$. With this notation we may identify X^* with $\ell^2(a^{-1})$ via $\xi\in\ell^2(a^{-1})\to\ell_\xi\in X^*$ where*

$$\ell_\xi(x):=\sum_{i=1}^\infty\xi_i x_i \text{ for all } x\in X.$$

Continuing the notation in Theorem 2.1, the measure $\mu:=\mu_a$ is a Gaussian measure on $(X:=X_a,\mathcal{B}:=\mathcal{B}_{X_a})$ and

$$\int_X e^{i\ell_\xi(x)}d\mu(x)=e^{-\frac12\|\xi\|_{\ell^2}^2}.$$

so μ is a Gaussian and $q(\ell_\xi,\ell_n)=(\xi,n)_{\ell^2}$.

8.1. Extras on ℓ^2 – Gaussian Measures.

Proposition 8.2. *Define for $\ell\in X^*$,*

$$x_\ell\equiv\int\ell(x)x\;d\mu(x).$$

If $\ell=\ell_\xi$, then $x_\ell=\int_X\sum_{i=1}^\infty\xi_i x_i x\;d\mu(x)$ and hence $(x_\ell)_i=\int\Sigma\xi_i x_i x_j\;d\mu(x)=\xi_i$. So $x_{\ell_\xi}=\xi$, and $X^*\xrightarrow{J}X$, $\quad J\ell_\xi:=\xi\in\ell^2$ and $(\xi,\eta)=q(\ell_\xi,\ell_n)=(\xi,n)_{\ell^2}$. Therefore $\overline{J(X^*)}^{(\cdot,\cdot)}=\ell^2$

Question: What happened to the Hilbert space ℓ^2 that μ was modeled on.

Some answers: Let $a\in\ell^\infty$ such that $\sum_{i=1}^\infty a_i<\infty$, $a_i>0$. For $w\in\ell^2(a)$ we have $\ell(x)=(w,x)_a$ is defined on $\ell^2(a)$. Now

$$Ee^{i\ell(x)}=\lim_{N\to\infty}Ee^{i\sum_{i=1}^N w_i x_i a_i}=\lim_{N\to\infty}e^{-\frac12\sum_1^N a_i^2w_i^2}=e^{-\frac12\sum_1^\infty a_i^2w_i^2}$$

Notice $\sum_{i=1}^\infty\ell(e_i)^2=\sum_{i=1}^\infty a_i^2w_i^2$

(1) Therefore $E[e^{i\ell}]=e^{-\frac12(\ell,\ell)_{H^*}}$ and $H=\ell^2$ has **reappeared**.
(2) Let again $\ell(x)=(w,x)_a$, $\quad w\in\ell^2(a)$, then $E(\ell(x)^2)=(\ell,\ell)_{H^*}$.
(3) Consider properties of μ under translations by $a\in\mathbb{R}^{\mathbb{N}}$.
Formally: $\frac{d\mu(x+a)}{d\mu(x)}=e^{-\left(x\cdot a+\frac12|a|^2\right)}$.
(4) $\ell(x)=(w,x)$ with $w\in\ell^2$, let $\ell_N(x)=\sum_{n\le N}w_nx_n$. Then $E\ell_N^2=\sum_{n\le N}w_n^2$, more generally

$$E(\ell_M-\ell_N)^2=\sum_{N<n\le M}w_n^2\to 0 \text{ as } M,N\to\infty.$$

Therefore $\ell_N\to n_w\in L^2(\mu)$. Write $n_w(x)=(w,x)$ define as an $L^2(\mu)$-valued random variable.

Proposition 8.3 (Cameron-Martin Type Theorem)**.** *Suppose $a\in\ell^2$ then $\mu(\cdot+a)\ll\mu$ and*

$$\frac{d\mu_a}{d\mu}=e^{-n_a(x)-\frac12|a|^2}. \tag{8.1}$$

Proof: By definition of μ_a, $\int fd\mu_a=\int f(\cdot-a)d\mu$ which follows by noting that

$$\text{when } f=1_A \text{ then } f(x-a)=1_A(x-a)=1_{A+a}(x).$$

So (8.1) is equivalent to

$$\int f(x-a)d\mu(x) = \int f(x)e^{-\overbrace{x\cdot a}^{n_a(x)}-\frac{1}{2}|a|^2}d\mu(x) \tag{8.2}$$

for all bounded measurable f. It suffice to check (8.2) for $f(x) = e^{\alpha\cdot x}, \alpha \in \ell_{fin}$. Now

$$\int e^{\alpha\cdot(x-a)}d\mu(x) = e^{-\alpha\cdot a}e^{+\frac{1}{2}|\alpha|^2},$$

while

$$\int e^{\alpha\cdot x}e^{-x\cdot a-\frac{1}{2}|a|^2}d\mu(x) = e^{\frac{1}{2}|\alpha-a|^2-\frac{1}{2}|a|^2} = e^{-\alpha\cdot a+\frac{1}{2}\ |\alpha|^2}$$

Q.E.D.

Theorem 8.4. *If $a \in \mathbb{R}^{\mathbb{N}} \setminus \ell^2$, then $\mu_a \perp \mu$.*

Proof: First notice that: $\mu_a \perp \mu$ iff $\|\mu_a - \mu\| = 2$. Let us compute $\|\mu_a - \mu\|$ making use of the notation, $z := e^{-x\cdot a/2+|a|^2/4}$:

$$\|\mu_a - \mu\| = \int \left|e^{-x\cdot a+\frac{1}{2}|a|^2} - 1\right| d\mu = \int |z-1|\ |z+1|d\mu$$
$$\geq \int |z-1|^2 d\mu = \int (z^2 - 2z + 1)d\mu = 2(1 - \int z d\mu).$$

Now

$$\int z d\mu = \int e^{-\frac{1}{2}(x\cdot a+\frac{1}{2}|a|^2)}d\mu$$
$$= \left(\int e^{-x\cdot\frac{a}{2}}d\mu\right)e^{-\frac{1}{4}|a|^2}$$
$$= e^{\frac{1}{2}|\frac{a}{2}|^2-\frac{1}{4}|a|^2} = e^{-\frac{1}{8}|a|^2}.$$

Therefore for $a \in \ell^2$,

$$\|\mu_a - \mu\| \geq 2(1 - e^{-\frac{1}{8}|a|^2}) = 2 \text{ if } a \notin \ell^2.$$

Q.E.D.

9. Classical Wiener Measure

Because of Proposition 7.3, the transpose $i^{tr} : W^* \to H^*$ is continuous as well. Hence by the Riesz theorem, for each $\phi \in W^*$ there exists a unique $h_\phi \in H$ such that $(h_\phi, \cdot)_H = \phi|_H$. Hence we may define an inner product q on W^* by $q(\phi, \psi) = (h_\phi, h_\psi)$ for all $\phi, \psi \in W^*$.

Theorem 9.1. *There exists a unique (Gaussian) measure μ on W such that*

$$\int_W e^{i\phi(x)}d\mu(x) = e^{-q(\phi)/2}, \tag{9.1}$$

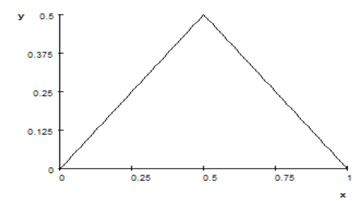
where $q(\phi, \psi) = (h_\phi, h_\psi)$ as described above.

$$\Psi(x) = x\,\text{Heaviside}(x) - (2x-1)\,\text{Heaviside}(x-1/2) + (x-1)\,\text{Heaviside}(x-1)$$

Proof: Let $\beta := \{\mathbf{1}\} \cup \{\psi_{kj} : 0 \leq k \text{ and } 0 \leq j < 2^k\}$ be the Haar basis for $L^2([0,1], d\lambda)$ introduced in Exercise **??** above and let $\Psi_0(t) := \int_0^t 1ds = t$ and $\Psi_{kj}(t) := \int_0^t \psi_{kj}(s)ds$ for $0 \leq k$ and $0 \leq j < 2^k$. Then $\tilde{\beta} = \{\Psi_0\} \cup \{\Psi_{kj} : 0 \leq k \text{ and } 0 \leq j < 2^k\}$ is an orthonormal basis for H which satisfies,

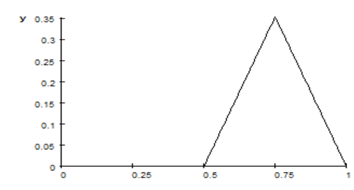
$$\|\Psi_0\|_\infty = 1 \text{ and } \|\Psi_{kj}\|_\infty = 2^{k/2}2^{-(k+1)} = \frac{1}{2}2^{-k/2}.$$

The following pictures shows the graphs of Ψ_{00}, $\Psi_{1,0}, \Psi_{1,1}, \Psi_{2,1}, \Psi_{2,2}$ and $\Psi_{2,3}$ respectively.

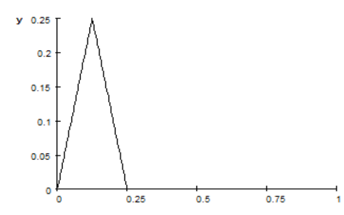


Plot of $\Psi_0, 0$.

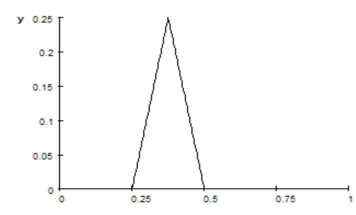
Plot of $\Psi_1 0$.



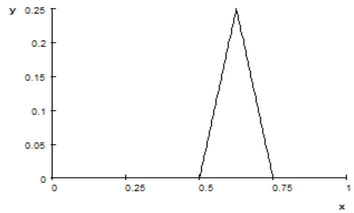
Plot of $\Psi_1 1$.



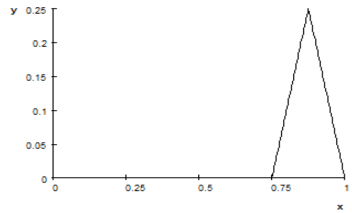
Plot of $\Psi_2 0$.



Plot of $\Psi_2 1$.



Plot of $\Psi_2 2$.



Plot of $\Psi_2 3$.

Because $\left\{\Psi_{kj} : 0 \le k \text{ and } 0 \le j < 2^k\right\}$ have disjoint supports,

$$\max_t \left|\sum_{j<2^k} x_j \Psi_{kj}(t)\right| \le \frac{1}{2} 2^{-\frac{k}{2}} \max\left\{|x_j| : j < 2^k\right\} \le \frac{1}{2} 2^{-\frac{k}{2}} \left(\sum_{j<2^k} |x_j|^p\right)^{\frac{1}{p}}$$

for any $p \in [1, \infty)$. Integrating this inequality, using Hölder's inequality, shows that

$$\int \left\|\sum_{j<2^k} x_{kj} \Psi_{kj}(t)\right\|_\infty d\mu(x) \le \frac{1}{2} 2^{-\frac{k}{2}} \left(\sum_{j<2^k} \int |x_{kj}|^p d\mu(x)\right)^{\frac{1}{p}}$$
$$= \frac{1}{2} 2^{-\frac{k}{2}} 2^{\frac{k}{p}} C_p$$

where

$$C_p = \int_{\mathbb{R}} |x|^p \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx < \infty.$$

Therefore

$$\int \sum_k \left\|\sum_{j<2^k} x_{kj} \Psi_{kj}(\cdot)\right\|_\infty d\mu(x) \le \frac{1}{Z} C_p \sum_{k=1}^\infty 2^{-\left(\frac{1}{2} - \frac{1}{p}\right)k} < \infty$$

providing $p > 2$. Letting

$$S_N = \sum_{k \le N} \left(\sum_{j<2^k} x_{kj} \Psi_{kj}(\cdot)\right)$$

we have it follows that $S(x) := \|\cdot\|_\infty\text{–}\lim_{N\to\infty} S_N(x)$ exists for μ – a.e. x. This completes the proof. Q.E.D.

Remark 9.2. We can use the above argument to prove more. Namely that Wiener measure lives on the space of paths which are α – Hölder continuous provided $\alpha < 1/2$.

Proof: To prove this refinement let $\alpha \in [0, 1]$ and consider

$$\Gamma_k(t) := \sum_{j<2^k} x_j \Psi_{kj}(t)$$

where $\{x_j\}_{j<2^k}$ are a fixed sequence of real numbers. By the mean value theorem and using the disjoint supports of $\{\Psi_{kj}\}$ and the fact that $\left\|\dot{\Psi}_{kj}\right\|_\infty = 2^{k/2}$, we find

$$|\Gamma_k(t) - \Gamma_k(s)| \le M 2^{k/2} |t - s|$$

where $M := \max\left\{x_j : j < 2^k\right\}$. Hence

$$\frac{|\Gamma_k(t) - \Gamma_k(s)|}{|t-s|^\alpha} \le 2^{k/2} M |t-s|^{1-\alpha}.$$

If $|t - s| \le 2^{-k}$ this gives

$$\frac{|\Gamma_k(t) - \Gamma_k(s)|}{|t-s|^\alpha} \le 2^{k/2} 2^{-k(1-\alpha)} M = 2^{-k(\frac{1}{2}-\alpha)} M$$

If $|t - s| \ge 2^{-k}$, recalling $\|\Gamma_k\|_\infty \le \frac{1}{2} 2^{-\frac{k}{2}} M$,we have

$$\frac{|\Gamma_k(t) - \Gamma_k(s)|}{|t-s|^\alpha} \le 2 2^{k\alpha} \frac{1}{2} 2^{-\frac{k}{2}} M = 2^{-k(\frac{1}{2}-\alpha)} M.$$

That is to say, $\|\Gamma_k\|_\alpha \le 2^{-k(\frac{1}{2}-\alpha)} M$ and working as above this implies

$$\sum_k \int \left\|\sum_{j<2^k} x_{kj} \Psi_{kj}(t)\right\|_\alpha d\mu(x) \le \sum_k \frac{1}{2} 2^{-k(\frac{1}{2}-\alpha)} \left(\sum_{j<2^k} \int |x_{kj}|^p d\mu(x)\right)^{\frac{1}{p}}$$
$$= \sum_k \frac{1}{2} C_p 2^{-k(\frac{1}{2}-\alpha)} 2^{\frac{k}{p}} < \infty$$

provided that

$$\frac{1}{2} - \alpha - \frac{1}{p} > 0$$

or $\alpha < 1/2 - 1/p$. Since we may take p as large as we like it follows that

$$\lim_{N\to\infty} \|S(x) - S_N(x)\|_\alpha = 0 \text{ for } \mu - \text{a.e. } x$$

and therefore $S(x) \in C^{0,\alpha}([0, 1], \mathbb{R})$ for μ – a.e. x. Q.E.D.

Lemma 9.3. *Let $\{b_s\}_{s\ge0}$ be a one dimensional Brownian motion. Then for any $T > 0$ and $\lambda \in [0, \pi^2/\left(4T^2\right))$;*

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2} \int_0^T b_s^2 ds\right)\right] = \cos^{-1/2}\left(\sqrt{\lambda} T\right) < \infty. \tag{9.2}$$

(All we really need for the qualitative result is to observe that, using Fernique's theorem,

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^T b_s^2 ds\right)\right] = \mathbb{E}\left[\exp\left(\frac{\lambda T^2}{2}\int_0^1 b_s^2 ds\right)\right] = 1 + \frac{\lambda T^2}{2}\mathbb{E}\left[\int_0^1 b_s^2 ds\right] + O\left(\lambda^2 T^4\right)$$
$$= 1 + \frac{\lambda T^2}{2}\int_0^1 s ds + O\left(\lambda^2 T^4\right) = 1 + \frac{\lambda T^2}{4} + O\left(\lambda^2 T^4\right)$$

for sufficiently small λT^2.)

Proof: When $T = 1$, simply follow the proof of [5, Eq. (6.9) on p. 472] with λ replaced by $-\lambda$. For general $T > 0$ observe that

$$\int_0^T b_s^2 ds = T\int_0^1 b_{tT}^2 dt \overset{d}{=} T^2 \int_0^1 b_t^2 dt$$

and therefore,

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^T b_s^2 ds\right)\right] = \mathbb{E}\left[\exp\left(\frac{\lambda T^2}{2}\int_0^1 b_s^2 ds\right)\right] = \cos^{-1/2}\left(\sqrt{\lambda T^2}\right)$$

provided that $\lambda \in [0, \pi^2/\left(4T^2\right)$.

Heuristic Proof.

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^1 b_s^2 ds\right)\right] = \frac{1}{Z}\int_{H([0,1],\mathbb{R})} \exp\left(-\frac{1}{2}\int_0^1 \left[\dot{x}^2(t) - \lambda x^2(t)\right] dt\right)\mathcal{D}x$$
$$= \frac{1}{Z}\int_{H([0,1],\mathbb{R})} \exp\left(-\frac{1}{2}\int_0^1 \left[-D^2 x(t) - \lambda x(t)\right] x(t)\, dt\right)\mathcal{D}x$$

where $D^2 = \frac{d^2}{dt^2}$ with Dirichlet boundary conditions at 0 and Neumann boundary conditions at 1. The eigenfunctions for D^2 are proportional to $\{\sin(l\pi x/2)\}_{l\in 2\mathbb{N}-1}$ with eigenvalues being $-l^2\pi^2/4$. Therefore,

$$\int_{H([0,1],\mathbb{R})} \exp\left(-\frac{1}{2}\int_0^1 \left[\dot{x}^2(t) - \lambda x^2(t)\right] dt\right)\mathcal{D}x = \frac{(2\pi)^{\dim H/2}}{\sqrt{\det\left(-D^2 - \lambda\right)}}$$
$$= \sqrt{\prod_{l \text{ odd}} \frac{2\pi}{l^2\pi^2/4 - \lambda}}$$

and Z is this expression with $\lambda = 0$. Hence we find,

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^1 b_s^2 ds\right)\right] = \sqrt{\prod_{l \text{ odd}} \frac{l^2\pi^2/4}{l^2\pi^2/4 - \lambda}} = \left(\prod_{l \text{ odd}}\left(1 - \frac{4\lambda}{l^2\pi^2}\right)\right)^{-1/2}$$

provided that $4\lambda < \pi^2$.

Rigorous Proof. Let $\{N_k\}_{k=1}^\infty$ be a sequence of i.i.d. normal random variables and $\{h_k\}_{k=1}^\infty$ be an orthonormal basis for $H = H([0,1],\mathbb{R})$. Then (see Proposition **??** below)

$$b_s := \sum_{k=1}^\infty N_k h_k(s)$$

is a standard one dimensional Brownian motion. Letting $a_k := \int_0^1 h_k^2(s)\, ds$, we then have

$$\int_0^1 b_s^2 ds = \sum_{k=1}^\infty N_k^2 a_k$$

and therefore that

$$\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^1 b_s^2 ds\right)\right] = \mathbb{E}\left[\exp\left(\frac{\lambda}{2}\sum_{k=1}^\infty N_k^2 a_k\right)\right]$$
$$= \prod_{k=1}^\infty \mathbb{E}\left[\exp\left(\frac{\lambda}{2}a_k N^2\right)\right] = \prod_{k=1}^\infty \frac{1}{\sqrt{1-\lambda a_k}}. \tag{9.3}$$

provided that $\lambda < 1/a_k$ for all k. Assuming this restriction on λ, the latter product is finite iff $\sum_{k=1}^\infty a_k < \infty$. But

$$\sum_{k=1}^\infty a_k = \sum_{k=1}^\infty \int_0^1 h_k^2(s)\, ds = \int_0^1 \sum_{k=1}^\infty h_k^2(s)\, ds = \int_0^1 s ds = 1/2 < \infty.$$

A calculus of variation exercises show

$$\sup_{h\neq 0} \frac{\int_0^1 h^2(s)\, ds}{\int_0^1 \dot{h}^2(s)\, ds} = \frac{4}{\pi^2}$$

and therefore $a_k \le 4/\pi^2$ for all k and the condition on λ becomes $\lambda < \pi^2/4$.

To get the more explicit expression for $\mathbb{E}\left[\exp\left(\frac{\lambda}{2}\int_0^1 b_s^2 ds\right)\right]$ let us observe that

$$\left\{h_l(t) = \frac{2\sqrt{2}}{\pi l}\sin\left(l\frac{\pi}{2}t\right)\right\}_{l \text{ odd}}$$

is an orthonormal basis for H. Indeed, $\left\{\dot{h}_l(t) = \sqrt{2}\cos\left(l\frac{\pi}{2}t\right)\right\}_{l \text{ odd}}$ is the orthonormal basis of eigenfunctions associated to d^2/dt^2 with Neumann boundary conditions at 0 and Dirichlet boundary conditions at 1. With this choice of basis we find

$$a_l = \frac{8}{\pi^2 l^2}\int_0^1 \sin^2\left(l\frac{\pi}{2}t\right) dt = \frac{4}{\pi^2 l^2}$$

which combined with Eq. (9.3) gives Eq. (**??**). Q.E.D.

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